

COARSE CATEGORIES I: FOUNDATIONS

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ABSTRACT. Following Roe and others (see, e.g., [9]), we (re)develop coarse geometry from the foundations, taking a categorical point of view. In this paper, we concentrate on the discrete case in which topology plays no role. Our theory is particularly suited to the development of the *Roe* (C^*-) algebras $C^*(X)$ and their K -theory on the analytic side; we also hope that it will be of use in the more strictly geometric/algebraic setting of controlled topology and algebra. However, we leave these topics to future papers.

Crucial to our approach are nonunital coarse spaces, and what we call *locally proper* maps; these are actually not new, being implicit in [2]. Our *coarse category* \mathbf{Crs} is a generalization of the usual one: its objects are (possibly nonunital) coarse spaces and its morphisms are (locally proper) coarse maps modulo *closeness*. \mathbf{Crs} is considerably richer than the usual coarse category of unital coarse spaces and proper coarse maps. As such, it has all nonzero limits and all colimits (all of which are easily constructed). We examine various other categorical issues. For example, \mathbf{Crs} does not have a terminal object, so we substitute a *termination functor*. This functor will be important in the development of exponential objects (i.e., “function spaces”) [13], and also leads to a notion of *quotient coarse spaces*. To connect our methods with the standard methods, we also examine the relationship between \mathbf{Crs} and the usual coarse category.

Finally we briefly discuss some basic examples and applications. Topics include *metric coarse spaces*, *continuous control* [1], metric and continuously controlled *coarse simplices*, σ -*coarse spaces* [5], and the relation between quotient coarse spaces and the K -theory of Roe algebras (which is of particular interest for continuously controlled coarse spaces).

INTRODUCTION

Coarse, or large-scale, geometry has long been studied in various guises, but most notably in the context of metric spaces. Most generically, a *coarse space* is a space together with some kind of large-scale structure (e.g., a metric modulo *quasi-isometry*; see Remark 5.1.3). A *coarse map* between coarse spaces is then a map which respects this structure (e.g., a large-scale Lipschitz map). Since the small-scale (i.e., the topology) is ignored, one can typically take coarse spaces to be *discrete*, replacing any nondiscrete space by some “coarsely dense” subset.

In recent decades, coarse ideas have played an important role in the study of infinite discrete groups using the methods of geometric group theory, especially in the work of Gromov and his followers (see, e.g., [8]). The most basic example here is that if Γ is a finitely generated group, then the word length metric on Γ is modulo quasi-isometry independent of the finite set of generators used in defining it.

Coarse ideas have also arisen in geometric topology, and more specifically controlled topology which primarily concerns itself with problems on the structure of manifolds. (We refer the reader to [31, Ch. 9] for a survey of the topic and for references.) In this setting, one is interested in “operations” (e.g., homotopies, surgeries) on spaces which respect some large-scale structure, i.e., are *controlled*. As before, one may take the large-scale structure to be given by a metric (i.e., *bounded control*). However, it is often more convenient to work with a coarser large-scale structure which is defined in purely topological terms (i.e., *continuous control*; see §5.2).

Controlled topology parallels the more classical theory for compact manifolds which relies on the use of algebraic invariants (e.g., algebraic K -theory). In controlled topology, one gets controlled versions of those invariants (in, e.g., *bounded* and *continuously controlled K -theory* [1, 22, 23]; see also [24]). By considering the fundamental group of a space, a key object of study in the study of homotopy invariants (e.g., the Novikov Conjecture on higher signatures), many of the problems of geometric topology are related back to geometric group theory.

On the analytic side, to any coarse space X , Roe has associated a C^* -algebra $C^*(X)$ (the *Roe algebra* of X), as well as various “(co)homology” groups, e.g., *coarse K -homology* $KX_\bullet(X)$. (For a good overview of this and the following, see [28].) On the other hand, one can also take the K -theory of $C^*(X)$; the Coarse Baum–Connes Conjecture is that a certain assembly map $KX_\bullet(X) \rightarrow K_\bullet(C^*(X))$ is an isomorphism, at least for suitably nice X .

The K -theory of Roe algebras arises in the index theory of elliptic operators on noncompact manifolds (on compact manifolds, the Roe algebra is just the compact operators and the results specialize to classical index theory). Indeed, historically it was the study of index theory on noncompact manifolds which led Roe to coarse geometry (see [27, 28]), and not the other way around. In this way, the analytic approaches to the Novikov Conjecture (starting with the work of Lusztig [12]) are again related to coarse geometry. (See [6] for a nice survey of the different approaches to the Novikov Conjecture.)

Roe’s coarse geometry. After originally developing coarse geometry in the metric context [25], Roe (and his collaborators) realized that one can define an abstract notion of *coarse space*, just as in small-scale geometry one has abstract *topological spaces*. Just as the passage from metric space to topological space forgets large-scale (metric) information, the passage from metric space to coarse space should forget small-scale information. But an abstract coarse space retains enough structure to perform the large-scale constructions which were previously done in the metric context (e.g., construct the Roe algebras, coarse K -homology, etc.).

A *coarse space* is a set X together with a *coarse structure*, which is a collection \mathcal{E}_X of subsets of $X^{\times 2} := X \times X$ (called the *entourages* of X) satisfying various axioms. When X is a (proper) metric space, \mathcal{E}_X consists of the subsets $E \subseteq X^{\times 2}$ such that

$$\sup\{d_X(x, x') : (x, x') \in E\} < \infty.$$

A subset $K \subseteq X$ is *bounded* if and only if $K^{\times 2}$ is an entourage of X ; when X is a metric space, K is bounded if and only if it is metrically bounded. If X is a discrete set, one typically axiomatically insists that the bounded subsets of X be finite (we call this the *properness axiom*; see Definition 1.3.1); more generally, if X is a topological space, the bounded subsets are required to be compact.

A set map $f: Y \rightarrow X$ is a *coarse map* if f is *proper* in the sense that the inverse image of any bounded subset of X is a bounded subset of Y and if f *preserves entourages* in the sense that $f^{\times 2}(F) := (f \times f)(F)$ is an entourage of X . In the metric case, f is a coarse map if it is metrically proper and “nonexpansive”.

There is an obvious notion of closeness for maps into a metric space: maps $f, f': Y \rightarrow X$ are *close* if

$$\sup\{d_X(f(y), f'(y)) : y \in Y\} < \infty.$$

This generalizes to the case when X is a general coarse space: f, f' are close if $(f \times f')(1_Y)$ is an entourage of X , where 1_Y is the diagonal set $\{(y, y) : y \in Y\}$.

Roe’s *coarse category* has coarse spaces as objects, and closeness classes of coarse maps as morphisms. (A coarse map is a *coarse equivalence* if it represents an isomorphism in the coarse category.) Coarse invariants are defined on this category, either as functions on the isomorphism classes of the coarse category (e.g., *asymptotic dimension*) or as functors from the coarse category to some other category (e.g., *coarse K-homology*). Though coarse invariants are the primary object of study in coarse geometry, the coarse category is rarely analyzed directly, and there is some confusion in the literature about what the coarse category is (some authors take its arrows to be actual coarse maps; we will call this the *pre-coarse category*).

There is an obvious “product coarse structure” on the cartesian (set) product $X \times Y$. The entourages are exactly the subsets of $(X \times Y)^{\times 2}$ which project to entourages of X and Y in the obvious way. However, this is not (usually) a product in the coarse category: the projection maps are not proper, unless both X and Y are finite/compact. This problem already arises in the category of proper metric spaces and proper maps (modulo closeness).

Remark. The above does *not* prove that X and Y do not have a product in the coarse category. Certain products (of infinite/noncompact coarse spaces) *do* exist in the coarse category; indeed, there is an infinite space X (namely the continuously controlled ray, or equivalently a countable set equipped with the *terminal*, i.e., “indiscrete”, coarse structure) such that the product of X with every countable coarse space exists (Remark 3.8.11). The above does not even prove that the “product coarse space” $X \times Y$, as defined above, is not a product of X and Y if equipped with suitable maps $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ (not the set projections).

Nonunital coarse spaces and locally proper maps. Metric spaces always yield *unital* coarse spaces, i.e., coarse spaces X such that $1_X := \{(x, x) : x \in X\}$ is an entourage. Though Roe defines nonunital coarse spaces, unitality is usually a

standing assumption, presumably since nonunital coarse spaces have no obvious use.

The major innovation of this paper is the following: We relax the requirement that coarse maps be proper, to a requirement that we call *locally properness*. Local properness is not new: it is actually included in Bartels's definition of "coarse map" [2, Def. 3.3]. When the domain is a unital coarse space, local properness is equivalent to ("global") properness (Corollary 1.6.5). However, when the domain is nonunital, we get many more coarse maps. Consequently, using nonunital coarse spaces, it becomes extremely easy to construct (nonzero) categorical products in the coarse category. Indeed, we can do much more.

Example. Suppose X' is a (closed) subspace of a proper metric space X , so that X' is itself a coarse space. There is an obvious *ideal* $\langle 1_{X'} \rangle_X$ of \mathcal{E}_X generated by $1_{X'}$ (see Definition 3.10.1). The coarse space $|X|_{\langle 1_{X'} \rangle_X}$ with underlying set X and coarse structure $\langle 1_{X'} \rangle_X$ is nonunital, unless X' is "coarsely dense" in X .

Define a (set) map $p: X \rightarrow X'$ which sends each $x \in X$ to a point $p(x)$ in X' closest to x . Then p is usually not proper, hence is not coarse as a map $X \rightarrow X'$. However, it is locally proper and coarse (in our generalized sense) as a map $|X|_{\langle 1_{X'} \rangle_X} \rightarrow X'$, and is actually a coarse equivalence. (We leave it to the reader to verify this, after locating the required definitions.)

For simplicity as well as for philosophical reasons, we only consider *discrete* coarse spaces; hence for us a map is (globally) proper if and only if the inverse image of any point is a finite set. If a map $f: Y \rightarrow X$ between coarse spaces is proper, then $f^{\times 2}$ is a proper map, and hence the restriction of $f^{\times 2}$ to any entourage $F \subseteq Y^{\times 2}$ of Y is proper. We take the latter as the definition of local properness: A map $f: Y \rightarrow X$ between coarse spaces (not necessarily unital) is *locally proper* if, for all entourages F of Y , the restriction $f^{\times 2}|_F: F \rightarrow X^{\times 2}$ is a proper map. There are a number of equivalent ways of defining local properness, the most intuitive of which is the following. For a nonunital coarse space, there is an obvious notion of *unital subspace*; a map is locally proper if and only if the restriction to every unital subspace of its domain is a proper map (Corollary 1.6.5).

When X is nonunital, we must modify the the definition of closeness, lest the identity map on X not be close to itself. We modify it in a simple way, now requiring that the domain also be a coarse space: Coarse maps $f, f': Y \rightarrow X$ (between possibly nonunital coarse spaces) are *close* if $(f \times f')(F)$ is an entourage of X for every entourage F of Y . After checking the usual things, we get our nonunital *coarse category*, whose objects are (possibly nonunital) coarse spaces and whose arrows are closeness classes of (locally proper) coarse maps.

Remark. Emerson–Meyer have defined a notion of σ -coarse spaces, coarse maps between such spaces, and an appropriate notion of closeness [5]. A σ -coarse space is just the colimit of an increasing sequence of unital coarse spaces. In fact, we show that the (pre)coarse category of discrete σ -coarse spaces is equivalent to a subcategory of our (pre)coarse category consisting of the σ -unital coarse

spaces (we do not examine the situation when one allows σ -coarse spaces to have nontrivial topologies).

Products, limits, etc. Let us see how to construct the product of coarse spaces X and Y in this category. We do so by putting a *nonunital* coarse structure on the set $X \times Y$. The entourages of the *coarse product* $X \times Y$ are the $G \subseteq (X \times Y)^{\times 2}$ such that:

- (i) the restricted projections $\pi_1|_G, \pi_2|_G: G \rightarrow X \times Y$ are proper maps (this is the aforementioned properness axiom);
- (ii) $\pi_X|_G: G \rightarrow X^{\times 2}$ and $\pi_Y|_G: G \rightarrow Y^{\times 2}$ are proper maps; and
- (iii) $(\pi_X)^{\times 2}(G)$ is an entourage of X and $(\pi_Y)^{\times 2}(G)$ is an entourage of Y .

One can then check that this is a product in our nonunital coarse category (indeed, it is a product in our nonunital *precoarse category*). We must emphasize that the coarse structure on the set product is crucial: If $*$ is a one-point coarse space, then $X \times * \cong X$ as a set, but unless X is bounded the coarse product $X \times *$ is *not* coarsely equivalent to X .

The above construction generalizes to all nonzero products (by nonzero product, we mean a product of a nonempty collection of spaces), including infinite products (Theorem 2.4.1 and Proposition 3.5.1). We will then proceed to examine equalizers in the nonunital coarse category, and discover that it has all equalizers of pairs of maps (Proposition 3.5.8). A standard categorical corollary is that the nonunital coarse category has all nonzero (projective) limits (Theorem 3.5.11). One can similarly analyze coproducts (i.e., sums or “disjoint unions”) and coequalizers, and get that the nonunital coarse category has all colimits, i.e., inductive limits (Theorem 3.7.6).

Terminal objects and quotients. For set theoretic reasons, the coarse category does not have a terminal object. (As we shall see in §3.13, one way of obtaining a terminal object is to restrict the cardinality of coarse spaces, though there is a better way to proceed. For most purposes, no such restriction is needed.) However, there is a plethora of coarse spaces which behave like terminal objects. The *terminal coarse structure* on a set X consists of the subsets of $X^{\times 2}$ which are both “row- and column-finite”; denote the resulting coarse space by $|X|_1$. A rather underappreciated fact about such coarse spaces is that, for any coarse space Y , *all* coarse maps $Y \rightarrow |X|_1$ are close to one another. An immediate categorical consequence of this is that, assuming that any such coarse map exists, the product of $|X|_1$ and Y in the (unital or nonunital) coarse category is just Y itself (Proposition 3.8.2).

In the unital coarse category, $X \mapsto |X|_1$ is a functor. In the nonunital coarse category, one must replace $|X|_1$ with a different coarse space, denoted $\text{Term}(X)$ (with $\text{Term}(X) = |X|_1$ for X unital), to obtain a functor. In an abelian category, one can define a quotient $X/f(Y)$ (for $f: Y \rightarrow X$) as push-out $X \amalg_Y 0$. This generalizes to any category with zero objects and push-out squares. We will see that in fact we can generalize this to the coarse setting, defining $X/[f](Y)$ to be the push-out $X \amalg_Y \text{Term}(Y)$ in the (nonunital) coarse category. (Indeed, one can do the same in the category of topological spaces, noting that there are two cases: “ $\text{Term}(X)$ ” is a one-point space if $X \neq \emptyset$ and the empty space otherwise.)

Applications. Our development of coarse geometry is a strict generalization of Roe’s, despite our assumption of discreteness (see §4). Most of the standard constructions in Roe’s coarse geometry (such as those mentioned above) generalize easily to our nonunital, locally proper version. (Note, however, that our theory does not encompass what one may call, following the language of [10, Ch. 12], the “uniform category” in which both the coarse structure and the topology are important. For example, Roe’s C^* -algebras $D^*(X)$, which are functorial for uniform maps, require a notion of *topological coarse space*. We defer this task to [14]; see Remark 4.2.3.) However, we will refrain from fully developing these applications in this paper. For the purposes of this paper, we briefly examine some things enabled by our generalization.

Having examined the coarse category from the categorical point of view, many standard constructions from topology transfer easily over to the coarse setting. For example, one obtains a notion of coarse simplicial complex. Of course, it is easy to deal with finite complexes explicitly in the unital coarse category. However, one result of having *all* colimits, including infinite ones, is that we actually obtain infinite coarse simplicial complexes. This should enable one to apply simplicial methods in coarse geometry.

Notes on history and references. The framework and terminology we use are essentially due to Roe and his collaborators (see [9, 25], in particular). Since our development differs in various details and in the crucial concept of local properness, and for the sake of completeness, we provide a complete exposition from basic principles; other, more standard, expositions include [9, 10, 28, 29]. In the basics, we do not claim much originality and most of the results will be known to those familiar with coarse geometry. However, in the context of locally proper maps, we have found certain methods of proof (in particular, the use of Proposition 1.2.2) to be particularly effective, and have emphasized the use of these methods. Thus our proofs of standard results may differ from the usual proofs.

We have endeavoured to provide reasonably thorough references. However, it is often unwieldy to provide complete data for things which have been generalized and refined over the years. In such cases, rather than providing references to the original definition and all the subsequent generalizations, we simply reference a work (often expository in nature) which provides the current standard definition; often, such definitions can be found in a number of places, such as the aforementioned standard expositions.

Organization. The rest of this paper is organized into five (very unequal) sections:

- §1: We define our basic framework of coarse structures, coarse spaces, and coarse maps, together with important results on local properness, and push-forward and pull-back coarse structures.

- §3: We consider the precoarse categories (and **PCrs** in particular) and their properties; the arrows in these categories are actual coarse maps. We discuss limits and colimits in these categories, as well as the relation between the general category **PCrs** and the subcategories of unital and/or connected coarse spaces.
- §3: We discuss the relation of closeness on coarse maps, establish basic properties of closeness, and consider the quotient coarse categories (**Crs** in particular). We show that **Crs** has all nonzero products and all equalizers, hence all nonzero limits. Similarly, it has all coproducts and all coequalizers, hence all colimits. We define the termination functor **Term**, and examine some of its properties; in particular, it provides “identities” for the product. We characterize the monic arrows of **Crs** and show that **Crs** has categorical images, and dually we do the same for epi arrows and coimages. We apply **Term**, together with push-outs, to define quotient coarse spaces. Finally, we discuss ways to “restrict” **Crs** to obtain subcategories with terminal objects.
- §4: We examine Roe’s formalization of coarse geometry, which allows coarse spaces to carry topologies, and the relation between the Roe coarse category and ours. In particular, we discuss how, given a “proper coarse space” (in the sense of Roe), one can functorially obtain a (discrete) coarse space (in our sense). We show that this gives a fully faithful functor from the Roe coarse category to **Crs**.
- §5: We give the basic examples of coarse spaces: those which come from proper metric spaces, and those which come from compactifications (i.e., continuously controlled coarse spaces). We define corresponding metric and continuously controlled coarse simplices, and indicate how one might then develop coarse simplicial theory. We compare Emerson–Meyer’s σ -coarse spaces to our nonunital coarse spaces (in the discrete case). Finally, we briefly examine the relation between quotients of coarse spaces, the K -theory of Roe algebras, and Kasparov K -homology.

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1. FOUNDATIONS OF COARSE GEOMETRY

Throughout this section, X , Y , and Z will be sets (sometimes with extra structure), and $f: Y \rightarrow X$ and $g: Z \rightarrow Y$ will be (set) maps. We denote the restriction of f to $T \subseteq Y$ by $f|_T: T \rightarrow X$. When $f(Y) \subseteq S \subseteq X$, we denote the range restriction of f to S by $f|_S: Y \rightarrow S$. Thus if $T \subseteq Y$ and $f(T) \subseteq S \subseteq X$, we have a restriction $f|_T^S: T \rightarrow S$.

1.1. Operations on subsets of $X \times X$. Much of the following can be developed in the more abstract context of groupoids [9], but we will refrain from doing so.

The basic object in question is the pair groupoid $X^{\times 2} := X \times X$. Recall that $X^{\times 2}$ has object set X and set of arrows $X \times X$. The map $X \hookrightarrow X^{\times 2}$ is $x \mapsto (x, x) =: 1_x$ for $x \in X$. The target and source maps are the projections $\pi_1, \pi_2: X^{\times 2} \rightarrow X$, respectively. For $x, x', x'' \in X$, composition is given by $(x, x') \circ (x', x'') := (x, x'')$ and the inverse by $(x, x')^{-1} := (x', x)$. Any set map $f: Y \rightarrow X$ induces a groupoid morphism

$$f^{\times 2} := f \times f: Y^{\times 2} \rightarrow X^{\times 2}$$

which in turn induces a map $\wp(Y^{\times 2}) \rightarrow \wp(X^{\times 2})$, again denoted $f^{\times 2}$, between power sets.

Definition 1.1.1 (operations on $\wp(X^{\times 2})$). For $E, E' \in \wp(X^{\times 2})$:

- (i) (*addition*) $E + E' := E \cup E'$;
- (ii) (*multiplication*) $E \circ E' := \{e \circ e' : e \in E, e' \in E', \text{ and } \pi_2(e) = \pi_1(e')\}$; and
- (iii) (*transpose*) $E^T := \{e^{-1} : e \in E\}$.

For $S \subseteq X$, put $1_S := \{1_x : x \in S\}$ (the *local unit* over S , or simply *unit* if $S = X$).

Proposition 1.1.2. For all $E \in \wp(X^{\times 2})$,

$$E \circ 1_S = (\pi_2|_E)^{-1}(S) \quad \text{and} \quad 1_S \circ E = (\pi_1|_E)^{-1}(S)$$

Remark 1.1.3. We will refrain from calling $E \circ E'$ a “product” to avoid confusion with cartesian/categorical products (e.g., $X \times Y$). The transpose E^T is often called the “inverse” and denoted E^{-1} ; we avoid this terminology and notation since it is somewhat deceptive (though, admittedly, also rather suggestive). Our units 1_X are usually denoted Δ_X (and called the diagonal, for obvious reasons); our terminology is more representative of the “algebraic” role played by the unit (and the local units) and avoids confusion with the (related) diagonal map $\Delta_X: X \rightarrow X \times X$ (where $X \times X$ is the cartesian/categorical product).

The operations of addition and composition make $\wp(X^{\times 2})$ into a semiring: addition is commutative with identity \emptyset , multiplication is associative with identity 1_X , multiplication distributes over addition, and $\emptyset \circ E = \emptyset = E \circ \emptyset$ for all E . Addition is idempotent in that $E + E = E$ for all E . Each 1_S is idempotent with respect to multiplication, i.e., $1_S \circ 1_S = 1_S$ for all S . The transpose is involutive, i.e., $(E^T)^T = E$ for all E , and, moreover, $(E + E')^T = E^T + (E')^T$, $(E \circ E')^T = (E')^T \circ E^T$, and $(1_S)^T = 1_S$, for all E, E' , and S .

Definition 1.1.4 (neighbourhoods and supports). For any $S \subseteq X$ and $E \in \wp(X^{\times 2})$, put

$$E \cdot S := \pi_1(E \circ 1_S) = \pi_1((\pi_2|_E)^{-1}(S)) \quad (\text{left } E\text{-neighbourhood of } S)$$

and

$$S \cdot E := \pi_2(1_S \circ E) = \pi_2((\pi_1|_E)^{-1}(S)) \quad (\text{right } E\text{-neighbourhood of } S).$$

We also call $E \cdot X = \pi_1(E)$ the *left support* of E and $X \cdot E = \pi_2(E)$ the *right support* of E .

Remark 1.1.5. The notations $N_E(S) := E_S := E[S] := E \cdot S$ and $E^S := S \cdot E$ are common, though our notation is hopefully more suggestive of the relation between $E \cdot S$, $S \cdot E$ and the previously defined operations.

Proposition 1.1.6. For all E, E' , and S :

$$\begin{aligned} (E + E') \cdot S &= E \cdot S \cup E' \cdot S & \text{and} & & S \cdot (E + E') &= S \cdot E \cup S \cdot E'; \\ E \circ 1_{E' \cdot S} &= E \circ E' \circ 1_S & \text{and} & & 1_{S \cdot E} \circ E' &= 1_S \circ E \circ E'; \\ (E \circ E') \cdot S &= E \cdot (E' \cdot S) & \text{and} & & S \cdot (E \circ E') &= (S \cdot E) \cdot E'; \quad \text{and} \\ E^T \cdot S &= S \cdot E & \text{and} & & S \cdot E^T &= E \cdot S. \end{aligned}$$

$\wp(X^{\times 2})$ and $\wp(X)$ are partially ordered by inclusion. All of the above “operations” are monotonic with respect to these partial orders.

Proposition 1.1.7. If $E_1, E_2, E'_1, E'_2 \in \wp(X^{\times 2})$ with $E_1 \subseteq E_2$ and $E'_1 \subseteq E'_2$, and $S_1, S_2 \subseteq X$ with $S_1 \subseteq S_2$, then:

$$\begin{aligned} E_1 + E'_1 &\subseteq E_2 + E'_2, & E_1 \circ E'_1 &\subseteq E_2 \circ E'_2, \\ (E_1)^T &\subseteq (E_2)^T, & 1_{S_1} &\subseteq 1_{S_2}, \\ E_1 \cdot S_1 &\subseteq E_2 \cdot S_2, & \text{and} & & S_1 \cdot E_1 &\subseteq S_2 \cdot E_2. \end{aligned}$$

1.2. Discrete properness. Since our coarse spaces are essentially discrete, for now we only discuss properness for maps between discrete sets.

Definition 1.2.1. A set map $f: Y \rightarrow X$ is *proper* if the inverse image $f^{-1}(K)$ of every finite subset $K \subseteq X$ is again finite.

If Y is itself a finite set, then any $f: Y \rightarrow X$ is automatically proper. We will use the following facts extensively (compare [3, §10.1 Prop. 5]).

Proposition 1.2.2. Consider the composition of (set) maps $Z \xrightarrow{g} Y \xrightarrow{f} X$:

- (i) If f and g are proper, then $f \circ g$ is proper.
- (ii) If $f \circ g$ is proper, then g is proper.
- (iii) If $f \circ g$ is proper and g is surjective, then f (and g) are proper.

Note that injectivity implies properness.

In (iii) above, the hypothesis that g be surjective can be weakened to the requirement that $Y \setminus g(Z)$ be a finite set. All restrictions (including range restrictions) of proper maps are again proper.

1.3. The properness axiom and coarse structures.

Definition 1.3.1. A set $E \in \wp(X^{\times 2})$ satisfies the *properness axiom* if the restricted target and source maps (i.e., projections) $\pi_1|_E, \pi_2|_E: E \rightarrow X$ (or, also restricting the ranges, $\pi_1|_E^{E \cdot X}, \pi_2|_E^{X \cdot E}$) are proper set maps.

Proposition 1.3.2. For $E \in \wp(X^{\times 2})$, the following are equivalent:

- (i) E satisfies the properness axiom;
- (ii) $E \circ 1_S$ and $1_S \circ E$ are finite for all finite $S \subseteq X$; and
- (iii) $E \circ E'$ and $E' \circ E$ are finite for all finite $E' \in \wp(X^{\times 2})$.

Proof. (i) \iff (ii): Immediate from $E \circ 1_S = (\pi_2|_E)^{-1}(S)$ (and symmetrically).

(ii) \iff (iii): The reverse implication is clear. For the forward implication, if E' is finite then $E' \cdot X$ is finite, and hence so too is

$$E \circ E' = E \circ E' \circ 1_X = E \circ 1_{E' \cdot X}$$

(and symmetrically). \square

Corollary 1.3.3. *If $E \in \wp(X^{\times 2})$ satisfies the properness axiom, then $E \cdot S$ and $S \cdot E$ are finite for all finite $S \subseteq X$.*

Proof. Use $E \cdot S := \pi_1(E \circ 1_S)$ (and similarly symmetrically). \square

Remark 1.3.4. The converse of the above Corollary holds since we are only considering pair groupoids: observe that

$$(\pi_1|_E)^{-1}(S) \subseteq S \times S \cdot E$$

(and similarly symmetrically). However, the converse does not hold in general for coarse structures on groupoids.

Proposition 1.3.5 (“algebraic” operations and the properness axiom). *If $E, E' \in \wp(X^{\times 2})$ satisfy the properness axiom, then $E + E'$, $E \circ E'$, E^T , and all subsets of E satisfy the properness axiom. Also, all singletons $\{e\}$, $e \in X^{\times 2}$, and hence all finite subsets of $X^{\times 2}$ satisfy the properness axiom, as does the unit 1_X .*

Proof. Clear, except possibly for $E \circ E'$; for this, use Proposition 1.3.2(iii) (and associativity of multiplication). \square

If T, T' are matrices over $X^{\times 2}$ (with values in some ring) are supported on $E, E' \in \wp(X^{\times 2})$ satisfying the properness axiom, then the product TT' is defined and has support contained in $E \circ E'$. The passage to rings of matrices motivates the following.

Definition 1.3.6. A coarse structure on X is a subset $\mathcal{E}_X \subseteq \wp(X^{\times 2})$ such that:

- (i) each $E \in \mathcal{E}_X$ satisfies the properness axiom;
- (ii) \mathcal{E}_X is closed under the operations of addition, multiplication, transpose, and the taking of subsets (i.e., if $E \subseteq E'$ and $E' \in \mathcal{E}_X$, then $E \in \mathcal{E}_X$); and
- (iii) for all $x \in X$, the singleton $\{1_x\}$ is in \mathcal{E}_X .

A coarse space is a set X equipped with a coarse structure \mathcal{E}_X on X . We denote such a coarse space by $|X|_{\mathcal{E}_X}$ or simply X . The elements of \mathcal{E}_X are called *entourages* (of \mathcal{E}_X or of X).

Example 1.3.7 (finite sets). If X is a finite set, then any coarse structure on X must be unital. Moreover, there is only one connected coarse structure on X , namely the power set of X .

Here are two natural coarse structures which exist on any set.

Definition 1.3.8. The *initial coarse structure* $\mathcal{E}_{|X|_0}$ on X is the minimum coarse structure on X . The *terminal coarse structure* $\mathcal{E}_{|X|_1}$ on a set X is the maximum coarse structure. (We denote the corresponding coarse spaces by $|X|_0$ and $|X|_1$, respectively.)

By Proposition 1.3.5, $\mathcal{E}_{|X|_1}$ simply consists of all the $E \in \wp(X^{\times 2})$ which satisfy the properness axiom. (Thus “ $E \in \mathcal{E}_{|X|_1}$ ” is a convenient abbreviation for “ $E \in \wp(X^{\times 2})$ satisfies the properness axiom”.) Any coarse structure on X is a subset of the terminal coarse structure (and obviously contains the initial coarse structure). More generally, we have the following.

Proposition 1.3.9. *The intersection of any collection of coarse structures on X (possibly infinite) is again a coarse structure on X .*

Definition 1.3.10. The coarse structure $\langle \mathcal{E}' \rangle_X$ on X generated by a subset $\mathcal{E}' \subseteq \mathcal{E}_{|X|_1}$ is the minimum coarse structure on X which contains \mathcal{E}' .

Of course, $\langle \mathcal{E}' \rangle_X$ is just the intersection of all the coarse structures on X containing \mathcal{E}' . Note that $\mathcal{E}_{|X|_0} = \langle \emptyset \rangle_X$; more concretely, $\mathcal{E}_{|X|_0}$ consists of all the finite local units 1_S , $S \subseteq X$ finite.

Given two subsets $\mathcal{E}', \mathcal{E}'' \subseteq \mathcal{E}_{|X|_1}$ (e.g., coarse structures on X), denote

$$\langle \mathcal{E}', \mathcal{E}'' \rangle_X := \langle \mathcal{E}' \cup \mathcal{E}'' \rangle_X.$$

Observe that $\langle \mathcal{E}', \mathcal{E}'' \rangle_X$ contains both $\langle \mathcal{E}' \rangle_X$ and $\langle \mathcal{E}'' \rangle_X$. We use similar notation given three or more subsets of $\mathcal{E}_{|X|_1}$ and, more generally, if $\{\mathcal{E}'_j : j \in J\}$ (J some index set) is a collection of subsets of $\mathcal{E}_{|X|_1}$,

$$\langle \mathcal{E}'_j : j \in J \rangle_X := \langle \bigcup_{j \in J} \mathcal{E}'_j \rangle_X.$$

One can describe the coarse structure generated by \mathcal{E}' rather more concretely.

Proposition 1.3.11. *If $\mathcal{E}' \subseteq \mathcal{E}_{|X|_1}$ contains all the singletons $\{1_x\}$, $x \in X$, and is closed under the “algebraic” operations of addition, multiplication, and transpose, then*

$$\langle \mathcal{E}' \rangle_X = \{E \subseteq \mathcal{E}' : E' \in \mathcal{E}'\}.$$

Corollary 1.3.12. *For any $\mathcal{E}' \subseteq \mathcal{E}_{|X|_1}$, $\langle \mathcal{E}' \rangle_X$ consists of the all subsets of the “algebraic closure” of the union*

$$\mathcal{E}' \cup \{\{1_x\} : x \in X\}.$$

Subsets of coarse spaces are naturally coarse spaces.

Definition 1.3.13. Suppose X is a coarse space and $X' \subseteq X$ is a subset. Then

$$\mathcal{E}_{X'} := \mathcal{E}_X|_{X'} := \mathcal{E}_X \cap \wp((X')^{\times 2})$$

is a coarse structure on X' , called the *subspace coarse structure*. Call $X' \subseteq X$ equipped with the subspace coarse structure a (coarse) *subspace* of X .

Example 1.3.14 (discrete metric spaces). Let (X, d) be a discrete, proper metric space. (X is metrically *proper* if closed balls of X are compact; thus X is discrete and proper if and only if every metrically bounded subset is finite.) The (d) -metric coarse space $|X|_d$ (or just $|X|$ for short) has as entourages the $E \in \mathcal{E}_{|X|_1} \subseteq \wp(X^{\times 2})$ such that

$$(1.3.15) \quad \sup\{d(x, x') : (x, x') \in E\} < \infty.$$

We may also allow $d(x, x') = \infty$ (for $x \neq x'$). In the senses defined below, $|X|_d$ is always unital but is connected if and only if $d(x, x') < \infty$ always. If $X' \subseteq X$,

then the metric coarse structure on X' obtained from the restriction of the metric d is just the subspace coarse structure $\mathcal{E}_{|X|_d}|_{X'}$.

1.4. Unitality and connectedness.

Definition 1.4.1. A coarse structure \mathcal{E}_X on X is *unital* if $1_X \in \mathcal{E}_X$. \mathcal{E}_X is *connected* if every singleton $\{e\}$, $e \in X^{\times 2}$, is an entourage of \mathcal{E}_X . A pair of points $x, x' \in X$ are *connected* (with respect to \mathcal{E}_X , or in the coarse space X) if $\{(x, x')\} \in \mathcal{E}_X$.

Most treatments of coarse geometry assume both unitality and connectedness, but we will assume neither. Connectedness is a relatively benign assumption (see §2.3), but *not* assuming unitality will be particularly crucial.

Remark 1.4.2. Connectedness in the general coarse groupoid case is more complicated, since there may be multiple arrows having the same target and source, and since a groupoid itself may not be connected. Let $\mathcal{E}_{\mathcal{G}}$ be a coarse structure on a groupoid \mathcal{G} . There are several possible notions of connectedness:

- (i) The obvious translation of the above to groupoids is to say that $\mathcal{E}_{\mathcal{G}}$ is (locally) *connected* if all singletons $\{e\}$ (e an arrow in the groupoid) are entourages of $\mathcal{E}_{\mathcal{G}}$.
- (ii) $\mathcal{E}_{\mathcal{G}}$ is *globally connected* if it is (locally) connected and \mathcal{G} is connected as a groupoid.

Objects x, x' are *connected* if *all* arrows e with target x and source x' yield entourages $\{e\}$. Then $\mathcal{E}_{\mathcal{G}}$ is (locally) connected if and only if *all groupoid-connected* pairs of objects are connected, and globally connected if and only if *all* pairs of objects are connected. But there is also a weaker notion of connectedness: x, x' are *weakly connected* if there is *some* arrow e with target x and source x' such that $\{e\}$ is an entourage.

- (iii) $\mathcal{E}_{\mathcal{G}}$ is (locally) *weakly connected* if all groupoid-connected objects x, x' are weakly connected.
- (iv) $\mathcal{E}_{\mathcal{G}}$ is *globally weakly connected* if it is (locally) weakly connected and \mathcal{G} is connected as a groupoid.

When \mathcal{G} is a pair groupoid (i.e., in our case), all the above notions coincide.

Proposition 1.4.3. *The terminal structure on any set X is always unital and connected.*

The intersection of unital coarse structures on a set X is again unital, and similarly for connected coarse structures. Thus, for any $\mathcal{E}' \subseteq \mathcal{E}_{|X|_1}$, there are *unital*, *connected*, and *connected unital* coarse structures on X generated by \mathcal{E}' . These can be described rather simply as

$$\begin{aligned} \langle \mathcal{E}' \rangle_X^U &:= \langle \mathcal{E}', \{1_X\} \rangle_X, \\ \langle \mathcal{E}' \rangle_X^C &:= \langle \mathcal{E}', \{\{e\} : e \in X^{\times 2}\} \rangle_X, \quad \text{and} \\ \langle \mathcal{E}' \rangle_X^{CU} &:= \langle \mathcal{E}', \{\{e\} : e \in X^{\times 2}\}, \{1_X\} \rangle_X, \end{aligned}$$

respectively.

Definition 1.4.4. The *initial unital*, *initial connected*, or *initial connected unital coarse structure* on a set X is the minimum coarse structure having the given property or properties, respectively. Denote the resulting coarse spaces by $|X|_0^U$, $|X|_0^C$, and $|X|_0^{CU}$, respectively.

Clearly, $\mathcal{E}_{|X|_0^U} = \langle \{1_X\} \rangle_X$, so a coarse structure on X is unital if and only if it contains $\mathcal{E}_{|X|_0^U}$. Similarly for the other properties. Note in particular that $\mathcal{E}_{|X|_0^C}$ consists of all the finite subsets of $X^{\times 2}$.

Remark 1.4.5. In the groupoid case, the intersection of (locally) connected coarse structures on a given groupoid (in the sense of Remark 1.4.2) is again (locally) connected, and so all of the above holds. However, the intersection of weakly connected coarse structures (see the same Remark) may not be weakly connected, so there may not be a minimum weakly connected coarse structure on a given groupoid.

We get an obvious notion of *unital subspace* of any coarse space X . Clearly, $X' \subseteq X$ is a unital subspace if and only if $1_{X'}$ is an entourage of X . (Bartels calls the set of unital subspaces of X the “domain of \mathcal{E}_X ” [2, Def. 3.2].) Slightly more is true.

Proposition 1.4.6. A subspace $X' \subseteq X$ of a coarse space X is unital if and only if it occurs as the left (or right) support of some entourage of X .

Proof. If X' is a unital subspace, then $X' = 1_{X'} \cdot X$. Conversely, if $X' = E \cdot X$ for some $E \in \mathcal{E}_X$, then $1_{X'} \subseteq E \circ E^T$ must be an entourage of X . \square

Similarly, we get a notion of *connected subspace* of X .

Definition 1.4.7. A (connected) *component* of a coarse space X is a maximal connected subspace of X .

Proposition 1.4.8. Any coarse space X is partitioned, as a set, into (a disjoint union of) its connected components.

We caution this decomposition is not necessarily a coproduct (in the coarse or precoarse category); see Corollary 2.5.4.

1.5. Local properness, preservation, and coarse maps. Recall that any (set) map $f: Y \rightarrow X$ induces a map (indeed, a groupoid morphism) $f^{\times 2}: Y^{\times 2} \rightarrow X^{\times 2}$. Insisting that f be proper is too strong a requirement when Y is a nonunital coarse space. We thus introduce the following weaker requirement.

Definition 1.5.1. A map $f: Y \rightarrow X$ is *locally proper* for $F \in \mathcal{E}_{|Y|_1}$ if $E := f^{\times 2}(F) \in \mathcal{E}_{|X|_1}$ and the restriction $f^{\times 2}|_F: F \rightarrow X^{\times 2}$ (or $f^{\times 2}|_F^E$) is a proper (set) map. If Y is a coarse space, then f is *locally proper* (for \mathcal{E}_Y) if it is locally proper for all $F \in \mathcal{E}_Y$.

Local properness only requires a coarse structure on the domain, so we cannot say that the composition of locally proper maps is again locally proper. Nonetheless, separating local properness from the following will be useful when we define push-forward coarse structures (below).

Definition 1.5.2. Suppose X is a coarse space. A map $f: Y \rightarrow X$ *preserves* $F \in \mathcal{E}_{|Y|_1}$ (with respect to \mathcal{E}_X) if $E := f^{\times 2}(F) \in \mathcal{E}_X$. If Y is also a coarse space, then f *preserves entourages* (of \mathcal{E}_Y , with respect to \mathcal{E}_X) if f preserves every $F \in \mathcal{E}_Y$.

Definition 1.5.3. Suppose X is a coarse space. A map $f: Y \rightarrow X$ is *coarse* for $F \in \mathcal{E}_{|Y|_1}$ if f is locally proper for F and if f preserves F . If Y is also a coarse space, then f is *coarse* (or is a *coarse map*) if f is coarse for every $F \in \mathcal{E}_Y$, i.e., if f is locally proper and preserves entourages.

Remark 1.5.4. The definition of “coarse map” is slightly redundant: If f preserves entourages, then $f^{\times 2}(F) \in \mathcal{E}_X \subseteq \mathcal{E}_{|X|_1}$ (which is one of the stipulations of local properness).

Proposition 1.5.5. Consider a composition of $Z \xrightarrow{g} Y \xrightarrow{f} X$, where X and Y are coarse spaces. If f, g are locally proper and g preserves entourages, then $f \circ g$ is locally proper.

Corollary 1.5.6. A composition of coarse maps is again a coarse map.

1.6. Basic properties of maps. We first concentrate on local properness.

Proposition 1.6.1. Suppose $f: Y \rightarrow X$ is a set map, $F \in \mathcal{E}_{|Y|_1}$, and $E := f^{\times 2}(F)$. The following are equivalent:

- (i) f is locally proper for F ;
- (ii) the restrictions $f|_{F \cdot Y}$ and $f|_{Y \cdot F}$ (or $f|_{F \cdot Y}^{E \cdot X}$ and $f|_{Y \cdot F}^{X \cdot E}$) of f to the left and right supports of F are proper; and
- (iii) $f^{-1}(S) \cdot F$ and $F \cdot f^{-1}(S)$ are finite for all finite $S \subseteq X$.

Proof. (We omit proofs of the symmetric cases.) Consider the diagram

$$\begin{array}{ccc} F & \xrightarrow{f^{\times 2}|_F^E} & E \\ \pi_1|_F^{F \cdot Y} \downarrow & & \pi_1|_E^{E \cdot X} \downarrow \\ F \cdot Y & \xrightarrow{f|_{F \cdot Y}^{E \cdot X}} & E \cdot X \end{array} \quad .$$

Observe the following: the above diagram commutes, i.e.,

$$\pi_1|_E^{E \cdot X} \circ f^{\times 2}|_F^E = f|_{F \cdot Y}^{E \cdot X} \circ \pi_1|_F^{F \cdot Y};$$

the two maps emanating from F are surjections; and $\pi_1|_F^{F \cdot Y}$ is proper. We now apply Proposition 1.2.2 several times.

(i) \Rightarrow (ii): $f^{\times 2}|_F^E$ and $\pi_1|_E^{E \cdot X}$ are proper, so their composition is proper. Since $\pi_1|_F^{F \cdot Y}$ is surjective, $f|_{F \cdot Y}$ is proper.

(ii) \Rightarrow (iii): $f|_{F \cdot Y}$ and $\pi_1|_F^{F \cdot Y}$ are proper, so their composition is proper. Then

$$\begin{aligned} f^{-1}(S) \cdot F &= \pi_2((\pi_1|_F)^{-1}(f^{-1}(S))) \\ (1.6.2) \quad &= \pi_2((f|_{F \cdot Y} \circ \pi_1|_F)^{-1}(S)) \end{aligned}$$

is finite if $S \subseteq X$ is finite.

(iii) \Rightarrow (i): By (1.6.2) and since $\pi_2|_F$ is proper, the composition $f|_{F \cdot Y}^{E \cdot X} \circ \pi_1|_F^{F \cdot Y}$ is proper. Hence $f^{\times 2}|_F^E$ is proper and, since $f^{\times 2}|_F^E$ is surjective, so is $\pi_1|_E^{E \cdot X}$. \square

Corollary 1.6.3. *If a set map $f: Y \rightarrow X$ is globally proper, then f is locally proper for any $F \in \mathcal{E}_{|Y|_1}$ (so f is locally proper for any coarse structure on Y).*

Proof. This follows from (iii) and Corollary 1.3.3. \square

Corollary 1.6.4. *If X is a coarse space and $X' \subseteq X$ is a subspace, then the inclusion of X' into X is a coarse map. Thus the restriction of any coarse map to a subspace is a coarse map.*

Proof. By definition of the subspace coarse structure, the inclusion map preserves entourages. The inclusion map is injective, hence (globally) proper, hence locally proper. \square

Corollary 1.6.5. *Suppose Y is a coarse space. A map $f: Y \rightarrow X$ is locally proper if and only if the restriction of f to every unital subspace of Y is proper. Thus, for Y unital, f is locally proper if and only if f is globally proper.*

Proof. This follows from (ii) and Proposition 1.4.6. \square

For (discrete) unital coarse spaces, our notion of “coarse map” is just the classical notion. It also follows that local properness of a map $f: Y \rightarrow X$ is a property which can be defined in terms of the unital subspaces of the coarse structure on Y . In particular, if f is locally proper, then f would also be locally proper for any coarse structure on Y (possibly larger than \mathcal{E}_Y) with the same unital subspaces.

Remark 1.6.6. One may take the *definition* of local properness to be the characterization of the above Corollary, i.e., define $f: Y \rightarrow X$ to be locally proper if f is proper on every unital subspace of Y (perhaps “unital properness” would be a more apt term). Indeed, this is the form in which local properness appears in Bartels’s definition of “coarse map” [2, Def. 3.3], and hence (modulo our coarse spaces not carrying topologies) our definition of “coarse map” is the same as Bartels’s. More generally, one could remove coarse structures entirely, and define local properness for sets equipped with families of supports (i.e., of unital subspaces). However, we will not do so since we are mainly concerned with coarse maps, for which Definition 1.5.1 is most convenient.

Corollary 1.6.7. *Coarse maps send unital subspaces to unital subspaces, i.e., if $f: Y \rightarrow X$ is a coarse map and $Y' \subseteq Y$ is a unital subspace, then the image $f(Y') \subseteq X$ is a unital subspace.*

Proposition 1.6.8 (“algebraic” operations and local properness). *If $f: Y \rightarrow X$ is locally proper for $F, F' \in \mathcal{E}_{|Y|_1}$, then f is locally proper for $F + F'$, $F \circ F'$, F^Γ , and all subsets of F . Also, f is locally proper for all singletons $\{e\}$, $e \in Y^{\times 2}$, hence is locally proper for $\mathcal{E}_{|Y|_0^c} \supseteq \mathcal{E}_{|Y|_0}$. (However, f is locally proper for the unit 1_Y if and only if f is globally proper.)*

Proof. The only nontrivial assertion is that f is locally proper for $F \circ F'$. By assumption, $f^{\times 2}(F), f^{\times 2}(F') \in \mathcal{E}_{|X|_1}$ and, since

$$f^{\times 2}(F \circ F') \subseteq f^{\times 2}(F) \circ f^{\times 2}(F'),$$

$f^{\times 2}(F \circ F')$ also satisfies the properness axiom, by Proposition 1.3.5. We have a commutative diagram

$$\begin{array}{ccc} F \circ F' & \xrightarrow{f^{\times 2}|_{F \circ F'}} & X^{\times 2} \\ \pi_1|_{(F \circ F') \cdot Y} \downarrow & & \downarrow \pi_1 \\ (F \circ F') \cdot Y & \xrightarrow{f|_{(F \circ F') \cdot Y}} & X \end{array} .$$

By the same Proposition, $F \circ F' \in \mathcal{E}_{|Y|_1}$, so $\pi_1|_{(F \circ F') \cdot Y}$ is proper. Since $(F \circ F') \cdot Y \subseteq F \cdot Y$ and $f|_{F \cdot Y}$ is proper by Proposition 1.6.1(ii), $f|_{(F \circ F') \cdot Y}$ is proper. Hence the composition

$$f|_{(F \circ F') \cdot Y} \circ \pi_1|_{(F \circ F') \cdot Y} = \pi_1 \circ f^{\times 2}|_{F \circ F'}$$

is proper, so $f^{\times 2}|_{F \circ F'}$ is proper by Proposition 1.2.2(ii). \square

Corollary 1.6.9. *If $f: Y \rightarrow X$ is locally proper for all $F \in \mathcal{E}' \subseteq \mathcal{E}_{|Y|_1}$, then f is locally proper for the coarse structure $\langle \mathcal{E}' \rangle_Y$ on Y generated by \mathcal{E}' (and for the connected coarse structure $\langle \mathcal{E}' \rangle_Y^{\mathbb{C}}$ generated by \mathcal{E}').*

Proof. This follows immediately from the Proposition and Corollary 1.3.12. \square

The same evidently does not hold for the unital (or connected unital) coarse structure generated by \mathcal{E}' .

We now state some parallel results for preservation of entourages. Combining these with the above results for local properness, we get parallel results for coarseness of maps.

Proposition 1.6.10. *Suppose X is a coarse space. If $f: Y \rightarrow X$ preserves $F, F' \in \mathcal{E}_{|Y|_1}$, then f preserves $F + F', F \circ F', F^T$, and all subsets of F . Also, f preserves all singletons $\{1_y\}$, $y \in Y$ (hence preserves $\mathcal{E}_{|Y|_0}$); if X is connected, f preserves all singletons $\{e\}$, $e \in Y^{\times 2}$ (hence preserves $\mathcal{E}_{|Y|_0^{\mathbb{C}}}$); and if X is unital, f preserves 1_Y .*

Proof. The only (slightly) nontrivial one is $F \circ F'$, for which ones uses

$$f^{\times 2}(F \circ F') \subseteq f^{\times 2}(F) \circ f^{\times 2}(F').$$

\square

Corollary 1.6.11. *Suppose X is a coarse space. If $f: Y \rightarrow X$ preserves $\mathcal{E}' \subseteq \mathcal{E}_{|Y|_1}$, then f preserves the coarse structure $\langle \mathcal{E}' \rangle_Y$ on Y generated by \mathcal{E}' . (If X is also connected, then f preserves $\langle \mathcal{E}' \rangle_Y^{\mathbb{C}}$; if X is unital, then f preserves $\langle \mathcal{E}' \rangle_Y^{\mathbb{U}}$; if X is both, then f preserves $\langle \mathcal{E}' \rangle_Y^{\mathbb{CU}}$.)*

Proposition 1.6.12 (“algebraic” operations and coarseness). *Suppose X is a coarse space. If $f: Y \rightarrow X$ is coarse for $F, F' \in \mathcal{E}_{|Y|_1}$, then f is coarse for $F + F', F \circ F', F^T$, and all subsets of F . Also, f is coarse for all singletons $\{1_y\}$, $y \in Y$ (hence is coarse for $\mathcal{E}_{|Y|_0}$); if X is connected, f is coarse for all singletons $\{e\}$, $e \in Y^{\times 2}$ (hence is coarse for $\mathcal{E}_{|Y|_0^{\mathbb{C}}}$); and if X is unital and f is proper, f is coarse for 1_Y .*

Corollary 1.6.13. *Suppose X and Y are coarse spaces, $\mathcal{E}' \subseteq \mathcal{E}_{|Y|_1}$, and $f: Y \rightarrow X$ is a set map.*

- (i) If $\mathcal{E}_Y = \langle \mathcal{E}' \rangle_Y$, then f is a coarse map if and only if f is coarse for all $F \in \mathcal{E}'$.
- (ii) If $\mathcal{E}_Y = \langle \mathcal{E}' \rangle_Y^C$, then f is a coarse map if and only if f is coarse for all $F \in \mathcal{E}'$ and all $\{e\}$, $e \in Y^{\times 2}$.
- (iii) If $\mathcal{E}_Y = \langle \mathcal{E}' \rangle_Y^U$, then f is a coarse map if and only if f is proper and f is coarse for (or preserves) all $F \in \mathcal{E}'$.
- (iv) If $\mathcal{E}_Y = \langle \mathcal{E}' \rangle_Y^{CU}$, then f is a coarse map if and only if f is proper and f is coarse for (or preserves) all $F \in \mathcal{E}'$ and all $\{e\}$, $e \in Y^{\times 2}$.

Note that requiring that f be coarse for all $\{e\}$, $e \in Y^{\times 2}$, is equivalent to requiring $f(y)$, $f(y')$ be connected for all $y, y' \in Y$.

If $f, f': Y \rightarrow X$ are (globally) proper maps, then certainly $f \times f': Y \times Y \rightarrow X \times X$ is proper. The same also holds locally, and this will be essential later.

Proposition 1.6.14. *If (set) maps $f, f': Y \rightarrow X$ are locally proper for $F \in \mathcal{E}_{|Y|_1}$, then:*

- (i) $E := (f \times f')(F) \subseteq X^{\times 2}$ satisfies the properness axiom; and
- (ii) the restriction $(f \times f')|_F^E: F \rightarrow E$ is a proper map.

Proof. Fix $F \in \mathcal{E}_{|Y|_1}$, put $E := (f \times f')(F)$, and consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{(f \times f')|_F^E} & E \\ \pi_1|_F^{F \cdot Y} \downarrow & & \pi_1|_E \downarrow \\ F \cdot Y & \xrightarrow{f|_{F \cdot Y}} & X \end{array} \quad .$$

The composition along the left and bottom is proper, and thus so is composition along the top and right. Consequently, $(f \times f')|_F^E$ is proper. Since $(f \times f')|_F^E$ is surjective, $\pi_1|_E$ is proper and similarly for $\pi_2|_E$. \square

We have the following “very” local analogue of Proposition 1.2.2. For a more general analogue, we will need push-forward coarse structures.

Proposition 1.6.15. *Consider the composition of (set) maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, supposing that $G \in \mathcal{E}_{|Z|_1}$ and putting $F := g^{\times 2}(G)$:*

- (i) If g is locally proper for G and f is locally proper for F , then $f \circ g$ is locally proper for G .
- (ii) If $f \circ g$ is locally proper for G , then g is locally proper for G .
- (iii) If $f \circ g$ is locally proper for G , then f is locally proper for F .

Proof. Put $E := f^{\times 2}(F)$. We apply Proposition 1.2.2 to the commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{g^{\times 2}|_G^F} & F & \xrightarrow{f^{\times 2}|_F^E} & E \\ \pi_1|_G \downarrow & & \pi_1|_F \downarrow & & \pi_1|_E \downarrow \\ Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \end{array} \quad .$$

(i) is clear. For (ii) and (iii): If $f \circ g$ is locally proper for G , then $\pi_1|_E$ and

$$(f \circ g)^{\times 2}|_G^E = f^{\times 2}|_F^E \circ g^{\times 2}|_G^F$$

are proper. By the latter, $g^{\times 2}|_G^E$ is proper. $g^{\times 2}|_G^E$ is surjective, so $f^{\times 2}|_F^E$ is also proper. Then

$$\pi_1|_E \circ f^{\times 2}|_F^E = f \circ \pi_1|_F$$

is proper, so $\pi_1|_F$ is proper. \square

1.7. Pull-back and push-forward coarse structures.

Definition 1.7.1. Suppose X is a coarse space. The *pull-back coarse structure* (of \mathcal{E}_X) on Y along (a set map) $f: Y \rightarrow X$ is

$$f^*\mathcal{E}_X := \{F \in \mathcal{E}_{|Y|_1} : f \text{ is coarse for } F\}.$$

By Proposition 1.6.12, $f^*\mathcal{E}_X$ is actually a coarse structure. If X is connected, then $f^*\mathcal{E}_X$ is connected. If X is unital and f is (globally) proper, then $f^*\mathcal{E}_X$ is unital. The following are clear.

Proposition 1.7.2. *If X is a coarse space and $f: Y \rightarrow X$ is a set map, then $f^*\mathcal{E}_X$ is the maximum coarse structure on Y which makes f into a coarse map.*

Corollary 1.7.3. *If $f: Y \rightarrow X$ is a coarse map, then f factors as a composition of coarse maps*

$$Y \xrightarrow{\beta} |Y|_{f^*\mathcal{E}_X} \xrightarrow{f} X,$$

where $\beta = \text{id}_Y$ and $f = f$ as set maps.

More generally, if $\{X_j: j \in J\}$ (J some index set) is a collection of coarse spaces and $\{f_j: Y \rightarrow X_j\}$ is a collection of set maps, then

$$\mathcal{E} := \bigcap_{j \in J} (f_j)^*\mathcal{E}_{X_j}$$

is the maximum coarse structure on Y which makes all the f_j into coarse maps. If Y is a coarse space and the $f_j: Y \rightarrow X_j$ are all coarse maps, then each f_j factors as a composition of coarse maps $f_j \circ \beta$ in the obvious way. Moreover, if all the X_j are connected, then \mathcal{E} is connected; if all the X_j are unital and all the f_j are (globally) proper, then \mathcal{E} is unital.

Definition 1.7.4. Suppose Y is a coarse space. The *push-forward coarse structure* (of \mathcal{E}_Y) on X along a *locally proper* map $f: Y \rightarrow X$ is

$$f_*\mathcal{E}_Y := \langle \{f^{\times 2}(F): F \in \mathcal{E}_Y\} \rangle.$$

We similarly define *unital*, *connected*, and *connected unital push-forward coarse structures*.

If Y is connected and f is surjective, then $f_*\mathcal{E}_Y$ is connected. Similarly, if Y is unital (hence f globally proper) and f is surjective, then $f_*\mathcal{E}_Y$ is unital.

Proposition 1.7.5. *If Y is a coarse space and $f: Y \rightarrow X$ is a locally proper map, then $f_*\mathcal{E}_Y$ is the minimum coarse structure on X which makes f into a coarse map.*

Corollary 1.7.6. *If $f: Y \rightarrow X$ is a coarse map, then f factors as a composition of coarse maps*

$$Y \xrightarrow{\tilde{f}} |X|_{f_*\mathcal{E}_Y} \xrightarrow{\alpha} X$$

where $\tilde{f} = f$ and $\alpha = \text{id}_X$ as set maps.

Of course, there are obvious unital, connected, and connected unital versions of the above. For the unital versions one needs f to be proper and Y should probably be unital; for the connected versions, Y should probably be connected.

More generally, if $\{Y_j: j \in J\}$ (J some index set) is a collection of coarse spaces and $\{f_j: Y_j \rightarrow X\}$ is a collection of locally proper maps, then

$$\mathcal{E} := \langle (f_j)_* \mathcal{E}_{Y_j} \rangle$$

is the minimum coarse structure on X which makes all the f_j into coarse maps. If X is a coarse space and the $f_j: Y_j \rightarrow X$ are all coarse maps, then each f_j factors as $\alpha \circ \tilde{f}_j$. Again, there are unital, connected, and connected unital versions of this.

Remark 1.7.7. We emphasize that whereas one can pull back coarse structures along *any* set map (or collection of set maps), one can only push forward coarse structures along *locally proper* maps. If one wants all the coarse structures to be unital (and take unital, possibly connected, push-forwards), then one evidently requires all maps to be (globally) proper.

It is easy to see what happens when one pushes a coarse structure forward and then pulls it back along the same map (or vice versa).

Proposition 1.7.8. *If Y is a coarse space and $f: Y \rightarrow X$ is a locally proper map, then $\mathcal{E}_Y \subseteq f^* f_* \mathcal{E}_Y$.*

Proof. f is coarse as a map $Y \rightarrow |X|_{f_*\mathcal{E}_Y}$. Applying Corollary 1.7.3, this map factors as $Y \xrightarrow{\beta} |Y|_{f^* f_* \mathcal{E}_Y} \rightarrow |X|_{f_* \mathcal{E}_Y}$ where β is the identity as a set map. \square

Proposition 1.7.9. *If X is a coarse space and $f: Y \rightarrow X$ is any set map, then $f_* f^* \mathcal{E}_X \subseteq \mathcal{E}_X$.*

Proof. Now f is coarse as a map $|Y|_{f^* \mathcal{E}_X} \rightarrow X$, to which we apply Corollary 1.7.6. \square

Using push-forward coarse structures (and Corollary 1.6.9), we can “restate” Proposition 1.6.15 as follows.

Proposition 1.7.10. *Consider the composition of (set) maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, where Z is a coarse space:*

- (i) *If g is locally proper and f is locally proper for the push-forward coarse structure $g_* \mathcal{E}_Z$ on Y , then $f \circ g$ is locally proper.*
- (ii) *If $f \circ g$ is locally proper, then g is locally proper.*
- (iii) *If $f \circ g$ is locally proper, then f is locally proper for the push-forward coarse structure $g_* \mathcal{E}_Z$ on Y .*

The above also hold with connected push-forward coarse structures in place of push-forward coarse structures. Also, that injectivity implies global properness implies local properness.

Remark 1.7.11. Applying the above Proposition with $Z := |Z|_1$ having the terminal coarse structure, we get (i) and (ii) of Proposition 1.2.2. If g is surjective, then the push-forward coarse structure $g_*\mathcal{E}_{|Z|_1}$ is the terminal coarse structure $\mathcal{E}_{|Y|_1}$ and we get (iii) as well.

2. THE PRECOARSE CATEGORIES

We now define several categories of coarse spaces, whose arrows are coarse maps, and examine their properties. These *precoarse categories* differ from the coarse categories, which are quotients of these categories (see §3).

2.1. Set and category theory. We will be unusually careful with our set and category theoretic constructions. The following can mostly be ignored safely, though will be needed eventually for rigorous, “canonical” constructions (e.g., when we consider sets of “all” modules over a coarse space).

Assuming the Grothendieck axiom that any set is contained in some universe, we first fix a universe \mathcal{U} (containing ω). *Small* (or \mathcal{U} -small) objects are elements of \mathcal{U} . A \mathcal{U} -category is one whose object set is a subset of \mathcal{U} . A (\mathcal{U}) -small category is one whose object set (hence morphism set and composition law) is in \mathcal{U} . A small category is necessarily a \mathcal{U} -category, but not vice versa. A \mathcal{U} -category in turn is \mathcal{U}^+ -small, where \mathcal{U}^+ denotes the smallest universe having \mathcal{U} as an element. A *locally small* \mathcal{U} -category is a \mathcal{U} -category whose Hom-sets $\text{Hom}(\cdot, \cdot)$ are all small.

Recall the notion of quotient categories (from, e.g., [17, Ch. II §8]): Given a category \mathcal{C} and an equivalence relation \sim on each Hom-set of \mathcal{C} , there is a *quotient category* \mathcal{C}/\sim and a *quotient functor* $\text{Quot}: \mathcal{C} \rightarrow \mathcal{C}/\sim$ satisfying the following universal property: For all functors $F: \mathcal{C} \rightarrow \mathcal{C}'$ (\mathcal{C}' any category, which can be taken to be \mathcal{U} -small if \mathcal{C} is \mathcal{U} -small) such that $f \sim f'$ (f, f' in some Hom-set of \mathcal{C}) implies $F(f) \sim F(f')$, there is a unique functor $F': \mathcal{C}/\sim \rightarrow \mathcal{C}'$ such that $F = F' \circ \text{Quot}$. Moreover, if the equivalence relation \sim is preserved under composition then, for all objects X, Y of \mathcal{C} , the set $\text{Hom}_{\mathcal{C}/\sim}(\text{Quot}(Y), \text{Quot}(X))$ is in natural bijection with the set of \sim -equivalence classes of $\text{Hom}_{\mathcal{C}}(Y, X)$.

As usual, **Set** denotes the category of small sets (and set maps). **Top** is the category of small topological spaces and continuous maps. Forgetful functors will be denoted by *Forget*, with the source and target categories (the latter often being **Set**) implied by context. For a category \mathcal{C} equipped with a forgetful functor to **Set**, we denote the full subcategory of \mathcal{C} of nonempty objects (i.e., those X with $\text{Forget}(X) \neq \emptyset$) by \mathcal{C}^\times .

For the most part, henceforth X, Y , and Z will be (small) coarse spaces, and $f: Y \rightarrow X$ and $g: Z \rightarrow Y$ coarse maps. $\mathbb{Z}_+ := \{n \in \mathbb{Z}: n \geq 0\}$ is the set of nonnegative integers and similarly $\mathbb{R}_+ := [0, \infty[$ is the set of nonnegative real numbers.

2.2. The precoarse categories.

Definition 2.2.1. The *precoarse category* **PCrs** has as objects all (small) coarse spaces and as arrows coarse maps. The *connected precoarse category* **CPCrs** is

full subcategory of **PCrs** consisting of the connected coarse spaces. Similarly define the *unital precoarse category* **UPCrs** and the *connected unital precoarse category* **CUPCrs**.

Remarks 2.2.2. In many ways, the category **CPCrs**[×] of nonempty connected coarse spaces, i.e., coarse spaces with exactly one connected component, is more natural. Observe that **CUPCrs** = **CPCrs** ∩ **UPCrs** is a full subcategory of the other three categories. (One might argue that the unital categories above are not the “correct” ones and further insist that the arrows in the unital categories should be “unit preserving”, i.e., surjective as set maps. However, the above unital categories are the usual ones used in coarse geometry; see §4 and especially Corollary 4.3.8.)

We will analyze various properties of the categories **PCrs** and **CPCrs** (which are better behaved than the others). In particular, we examine limits and colimits in these categories, which include as special cases products and coproducts, equalizers and coequalizers, and terminal and initial objects. (We use the standard terminology from category theory, topology, etc.: limits are also called “projective limits” or “inverse limits” and colimits are called “inductive limits” or “direct limits”, though “direct limits” are often more specifically filtered colimits.)

Let us first recall some standard terminology (see, e.g., [17]). Let \mathcal{C} be a category and suppose $\mathcal{F}_X: \mathcal{J} \rightarrow \mathcal{C}$ (\mathcal{J} a small, often finite, category) is a functor. A *cone* $v: X \rightarrow \mathcal{F}_X$ to \mathcal{F}_X consists of an $X \in \text{Obj}(\mathcal{C})$ and arrows $v_j: X \rightarrow X_j := \mathcal{F}_X(j)$, $j \in \text{Obj}(\mathcal{J})$, such that the triangles emanating from X commute. A *limit* in \mathcal{C} for \mathcal{F}_X is given by a cone $X \rightarrow \mathcal{F}_X$ which is universal, i.e., a *limiting cone*. Limits of \mathcal{F}_X in \mathcal{C} are unique up to natural isomorphism. Thus we will sometimes follow the customary abuses of referring to *the* limit of \mathcal{F}_X and of referring to the object X (often denoted $\text{Lim } \mathcal{F}_X$) as the limit with the v_j understood. A functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ *preserves limits* if whenever $v: X \rightarrow \mathcal{F}_X$ is a limiting cone in \mathcal{C} , $F \circ v: F(X) \rightarrow F \circ \mathcal{F}_X$ is limiting in \mathcal{C}' . Dually, one has *cones from* \mathcal{F}_Y , *colimits*, *colimiting cones*, and functors which *preserve colimits*. All limits and colimits considered will be small. In particular, the category \mathcal{J} and functors \mathcal{F}_X and \mathcal{F}_Y will be small.

First, we examine the relation between **PCrs** and **CPCrs**.

2.3. PCrs versus CPCrs. Below, I will always denote the inclusion **CPCrs** \hookrightarrow **PCrs**. Note that I is fully faithful.

Definition 2.3.1. $\text{Conn}: \mathbf{PCrs} \rightarrow \mathbf{CPCrs}$ is the functor defined as follows:

- (i) For a coarse space X , $\text{Conn}(X)$ is just X as a set, but with the *connected* coarse structure $\langle \mathcal{E}_X \rangle_X^{\mathcal{C}}$ generated by \mathcal{E}_X .
- (ii) For a coarse map $f: Y \rightarrow X$, $\text{Conn}(f): \text{Conn}(Y) \rightarrow \text{Conn}(X)$ is the same as a set map as f (which is coarse by Corollary 1.6.13).

The following is clear.

Proposition 2.3.2. $\text{Conn} \circ I$ is the identity functor on **CPCrs**.

Proposition 2.3.3. $\text{Conn}: \mathbf{PCrs} \rightarrow \mathbf{CPCrs}$ is left adjoint to the inclusion functor.

The counit maps $Y \rightarrow I(\text{Conn}(Y))$, $Y \in \text{Obj}(\mathbf{PCrs})$, of the above adjunction are just the identities as set maps. The unit maps $X = \text{Conn}(I(X)) \rightarrow X$, $X \in \text{Obj}(\mathbf{CPCrs})$, are the identity maps.

Proof. Since $\text{Conn} \circ I$ is the identity, Conn induces natural maps

$$\text{Hom}_{\mathbf{PCrs}}(Y, I(X)) \rightarrow \text{Hom}_{\mathbf{CPCrs}}(\text{Conn}(Y), X)$$

(for Y possibly disconnected and X connected), which are clearly bijections. \square

Corollary 2.3.4. $I: \mathbf{CPCrs} \hookrightarrow \mathbf{PCrs}$ preserves limits and $\text{Conn}: \mathbf{PCrs} \rightarrow \mathbf{CPCrs}$ preserves colimits. Moreover, if $\mathcal{F}: \mathcal{J} \rightarrow \mathbf{CPCrs}$ is a functor and v is a limiting cone to (or colimiting cone from) $I \circ \mathcal{F}$ in \mathbf{PCrs} , then $\text{Conn} \circ v$ is a limiting cone to (or colimiting cone from, respectively) $\mathcal{F} = \text{Conn} \circ I \circ \mathcal{F}$ in \mathbf{CPCrs} .

Proof. See, e.g., [17, Ch. V §5] or [30, 16.4.6] for the first statement, and [30, 16.6.1] for the second. \square

2.4. Limits in the precoarse categories.

Theorem 2.4.1. \mathbf{PCrs} has all nonzero limits (i.e., limits of functors $\mathcal{J} \rightarrow \mathbf{PCrs}$ for \mathcal{J} nonempty). Moreover, the forgetful functor $\text{Forget}: \mathbf{PCrs} \rightarrow \mathbf{Set}$ preserves limits, and the limits of connected coarse spaces are connected. Consequently, the same hold with \mathbf{CPCrs} in place of \mathbf{PCrs} .

It is actually easy to see that $\text{Forget}: \mathbf{PCrs} \rightarrow \mathbf{Set}$ preserves limits: Forget is naturally equivalent to the covariant Hom-functor $\text{Hom}_{\mathbf{PCrs}}(*, \cdot): \mathbf{PCrs} \rightarrow \mathbf{Set}$, where $*$ is any one-point coarse space, and thus preserves limits (see, e.g., [17, Ch. V §4 Thm. 1]). Since I do not know a similar argument for colimits, let us proceed in ignorance of this.

Proof. Recall that \mathbf{Set} has all limits. Given $\mathcal{F}_X: \mathcal{J} \rightarrow \mathbf{PCrs}$, fix a limiting set cone $v: X \rightarrow \text{Forget} \circ \mathcal{F}_X$, so that X is a set and $v_j: X \rightarrow X_j := \mathcal{F}_X(j)$, $j \in \text{Obj}(\mathcal{J})$, are set maps. It suffices to put a coarse structure on X so that we get a limiting cone $v: X \rightarrow \mathcal{F}_X$ in \mathbf{PCrs} (with X connected if all the X_j are connected).

We need all the $v_j: X \rightarrow X_j$ to become coarse maps. Taking the coarse structure on X to be the intersection

$$\mathcal{E}_X := \bigcap_{j \in \text{Obj}(\mathcal{J})} (v_j)^* \mathcal{E}_{X_j}$$

of pull-back coarse structures clearly makes this so. (Since pull-backs of connected coarse structures are connected and intersections of connected coarse structures are connected, \mathcal{E}_X is connected if all the X_j are.) Since Forget is faithful, $v: X \rightarrow \mathcal{F}_X$ is a cone in \mathbf{PCrs} . We must show that it is universal.

Suppose $\mu: Y \rightarrow \mathcal{F}_X$ is another cone in \mathbf{PCrs} . Applying Forget , we get a cone $\mu: Y \rightarrow \text{Forget} \circ \mathcal{F}_X$ in \mathbf{Set} (properly written $\text{Forget} \circ \mu: \text{Forget}(Y) \rightarrow \text{Forget} \circ \mathcal{F}_X$). Since v is universal in \mathbf{Set} , there is a set map $t: Y \rightarrow X$ such that $\mu = v \circ t$ as cones in \mathbf{Set} . We must show that t is actually a coarse map (uniqueness is clear).

First, since \mathcal{J} is nonzero, there is some object $j_0 \in \text{Obj}(\mathcal{J})$; then $\mu_{j_0} = \nu_{j_0} \circ t$ (as set maps) is locally proper, so t is locally proper (Proposition 1.7.10(ii)). Next, for each $j \in \text{Obj}(\mathcal{J})$ and $F \in \mathcal{E}_Y$, ν_j is coarse for $E := t^{\times 2}(F)$ (which is in $\mathcal{E}_{|X|_1}$, by local properness) and hence $E \in (\nu_j)^* \mathcal{E}_X$. Since $\mu_j = \nu_j \circ t$ is locally proper for F , ν_j is locally proper for E (Proposition 1.6.15(ii)), and also ν_j clearly preserves E .

For **CPCrs**, the assertions follow from Corollary 2.3.4. \square

The above proof gives a rather concrete description of limits in **PCrs** (and **CPCrs**), and in particular of products. The product $\mathbf{PCrs}\text{-}\prod_{j \in J} X_j$ in **PCrs** (or in **CPCrs**) is just the set product (i.e., cartesian product) $X := \mathbf{Set}\text{-}\prod_{j \in J} X_j$ together with the entourages of $|X|_1$ which project properly to entourages of all the X_j .

The “nonzero” stipulation in Theorem 2.4.1 is necessary.

Proposition 2.4.2. *For each coarse space X , there exists a (nonempty) connected, unital coarse space Y such that there is no coarse map $Y \rightarrow X$.*

Proof. Given X , take $Y := |Y|_1$ to be an infinite set with cardinality strictly greater than the cardinality of X , equipped with the terminal coarse structure (which is connected and unital), e.g., $Y := |\wp(X) \sqcup \mathbb{N}|_1$. Then no locally proper map $Y \rightarrow X$ exists, since no globally proper map $Y \rightarrow X$ exists and Y is unital (see Corollary 1.6.5). (Note that the cardinality of sets in our universe \mathcal{U} is bounded above by some cardinal, namely by $\#\mathcal{U}$, but no element of \mathcal{U} has this cardinality.) \square

Corollary 2.4.3. *None of the precoarse categories (**PCrs**, **CPCrs**, **CPCrs** $^\times$, **UPCrs**, and **CUPCrs**) has a terminal object.*

The failure of existence of terminal objects in the precoarse categories is not just a failure of uniqueness of maps, but more seriously of existence. Thus we will also get the following on the coarse categories (which are quotients of the precoarse categories).

Corollary 2.4.4. *No quotient of any of the above precoarse categories has a terminal object.*

It is straightforward to show that the inclusion $\mathbf{CPCrs}^\times \hookrightarrow \mathbf{CPCrs}$ preserves limits, and moreover that a nonzero limit exists in \mathbf{CPCrs}^\times if and only if the corresponding set limit is nonempty (but the example below shows that \mathbf{CPCrs}^\times does *not* have all nonzero limits). On the other hand unitality poses a fatal problem: The forgetful functor $\mathbf{UPCrs} \rightarrow \mathbf{Set}$ still preserves limits, so a (nonzero) limit in **UPCrs** can only exist when all the maps in the corresponding limiting set cone are proper (but this is often not the case, e.g., in the case of products).

Example 2.4.5. Let $X := |\mathbb{Z}_+|_1$ (which is connected and nonempty), $f := \text{id}_X: X \rightarrow X$ be the identity, and define $g: X \rightarrow X$ by $g(x) := x + 1$. Then the equalizer of f and g in **PCrs** is the empty set.

To get ahead of ourselves (see §3), note that though f and g are *close*, the equalizer of f and itself (which is just X mapping identically to itself) is not *coarsely*

equivalent to the equalizer of f and g . Indeed, one can obtain other inequivalent equalizers: e.g., the equalizer of $h: X \rightarrow X$ where $h(x) := \min\{0, x - 1\}$ (which also close to f) and f is $\{0\}$ (including into X). On the other hand, in the quotient coarse category \mathbf{Crs} , $[f] = [g] = [h]$ so the equalizer of any pair of these maps is X . Since limits in \mathbf{PCrs} are not *coarsely invariant*, they are of limited interest.

We also see that the quotient functor $\mathbf{PCrs} \rightarrow \mathbf{Crs}$ does not preserve equalizers, hence does not preserve limits. However, it *does* preserve products. Using this and a method parallel to the one employed in the proof of Theorem 2.4.1, we will show that \mathbf{Crs} also has all nonzero limits (which will, by definition, be coarsely invariant).

We will use products extensively. We take this opportunity to mention several canonical coarse maps which arise due to the existence of (nonzero) products (all objects are coarse spaces and arrows coarse maps):

- (i) For any X and Y , there are *projection maps* $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$.
- (ii) For any X , there is a *diagonal map* $\Delta_X: X \rightarrow X \times X$.
- (iii) For $f: Y \rightarrow X$ and $f': Y' \rightarrow X'$, there is a *product map* $f \times f': Y \times Y' \rightarrow X \times X'$.

The above can all be generalized to larger (even infinite) products.

2.5. Colimits in the precoarse categories.

Theorem 2.5.1. *A colimit exists in \mathbf{PCrs} if and only if all the maps from a corresponding colimiting set cone are locally proper. Moreover, the forgetful functor $\text{Forget}: \mathbf{PCrs} \rightarrow \mathbf{Set}$ preserves colimits. The same hold with \mathbf{CPCrs} in place of \mathbf{PCrs} .*

Proof. This proof is basically dual to the proof of Theorem 2.4.1, only with the added onus of showing the “only if”. The reason for the local properness requirement is that coarse structures can only be pushed forward along locally proper maps (whereas they can be pulled back along all maps).

Recall that \mathbf{Set} has all colimits. Given $\mathcal{F}_Y: \mathcal{J} \rightarrow \mathbf{PCrs}$, fix a colimiting set cone $\nu: \text{Forget} \circ \mathcal{F}_Y \rightarrow Y$, so that Y is a set and $\nu_j: Y_j := \mathcal{F}_Y(j) \rightarrow Y$, $j \in \text{Obj}(\mathcal{J})$, are set maps. Suppose all of the ν_j are locally proper. Taking the coarse structure on Y to be

$$\mathcal{E}_Y := \langle (\nu_j)_* \mathcal{E}_{Y_j} : j \in \text{Obj}(\mathcal{J}) \rangle_Y,$$

we clearly get a cone $\nu: \mathcal{F}_Y \rightarrow Y$ in \mathbf{PCrs} ; we must prove that it is universal.

Suppose $\mu: \mathcal{F}_Y \rightarrow X$ is another cone in \mathbf{PCrs} . Then there is a canonical set map $t: Y \rightarrow X$ such that $\mu = t \circ \nu$ as cones in \mathbf{Set} . We must show that t is coarse (again uniqueness is clear). Entourages $(\nu_j)^{\times 2}(F)$, $F \in \mathcal{E}_{Y_j}$, $j \in \text{Obj}(\mathcal{J})$, generate \mathcal{E}_Y . t is locally proper for each such entourage (using $\mu_j = t \circ \nu_j$ and Proposition 1.6.15(iii)) and clearly preserves each such entourage. Thus t is coarse, as required.

If the ν_j are *not* all locally proper, we must show that \mathcal{F}_Y does not have a colimit (in \mathbf{PCrs}); in fact, we show something stronger, that there is no cone from \mathcal{F}_Y in \mathbf{PCrs} . We proceed by contradiction, so suppose that ν_{j_0} is not locally proper ($j_0 \in \text{Obj}(\mathcal{J})$ fixed) and suppose $\mu: \mathcal{F}_Y \rightarrow X$ is a cone in \mathbf{PCrs} . Again there must

be a set map $t: Y \rightarrow X$ such that $\mu = t \circ \nu$ as set cones. But then $\mu_{j_0} = t \circ \nu_{j_0}$ is locally proper, which implies that ν_{j_0} is locally proper (Proposition 1.7.10(ii)) which is a contradiction.

To get the asserted colimits in **CPCrs**, simply apply Corollary 2.3.4. To show that **CPCrs** has no more colimits than **PCrs** (i.e., $\mathcal{F}_Y: \mathcal{J} \rightarrow \mathbf{CPCrs}$ has a colimit in **CPCrs** only if $I \circ \mathcal{F}_Y: \mathcal{J} \rightarrow \mathbf{PCrs}$ has a colimit in **PCrs**), it is probably simplest to modify the above proof. \square

The following are clear.

Corollary 2.5.2. ***PCrs** and **CPCrs** have all coproducts.*

Corollary 2.5.3. *The empty coarse space is the (unique) initial object in **PCrs** and in **CPCrs**.*

Corollary 2.5.4. *Any coarse space with only finitely many connected components is (isomorphic in **PCrs** to) the coproduct in **PCrs** of its connected components.*

The above Corollary does not necessarily hold for coarse spaces with infinitely many connected components. One may say, more generally, that any coarse space whose unital subspaces have only finitely many connected components is the coproduct of its connected components.

We get concrete descriptions of coproducts in **PCrs** and in **CPCrs**. The coproduct $\mathbf{PCrs}\text{-}\coprod_{j \in \mathcal{J}} Y_j$ in **PCrs** is just the set coproduct (i.e., disjoint union) $Y := \mathbf{Set}\text{-}\coprod_{j \in \mathcal{J}} Y_j$ with entourages finite unions of entourages of the Y_j (included into Y). The corresponding coproduct in **CPCrs** is the same as a set, but one may also take an additional union with an arbitrary finite subset of $Y^{\times 2}$.

The inclusion $\mathbf{CPCrs}^\times \hookrightarrow \mathbf{CPCrs}$ preserves colimits. \mathbf{CPCrs}^\times does not have a zero colimit (i.e., initial object), but otherwise has a colimit if the corresponding colimit exists in **CPCrs**, in which case the two colimits coincide; note that a nonzero colimit of nonempty sets is nonempty. Unitality does not pose a problem for colimits: Theorem 2.5.1 also holds with **UPCrs** in place of **PCrs** (and **CUPCrs** in place of **CPCrs**). In the proof, one simply takes the unital coarse structure

$$\langle (v_j)_* \mathcal{E}_{Y_j} : j \in \text{Obj}(\mathcal{J}) \rangle_Y^{\cup}$$

instead. Of course, in the unital cases, one may substitute “(globally) proper” for “locally proper”.

The “locally proper” hypothesis is necessary, as the following shows.

Example 2.5.5. Let $X := |\mathbb{Z}_+|_1$, $f: X \rightarrow X$ be the identity, and define $g: X \rightarrow X$ by $g(x) := \min\{0, x - 1\}$. The coequalizer of f and g in **Set** is the one-point set $*$; since X is unital, f and g do not have a coequalizer in **PCrs**.

Again, to get ahead of ourselves, we see that coequalizers in **PCrs** are not coarsely invariant. Even though f is close to g and the coequalizer of f and itself is just X , f and g do not have a coequalizer in **PCrs**. In the quotient category **Crs**, there are no problems: the coequalizer of $[f]$ and $[g]$ is X , as expected.

The quotient functor $\mathbf{PCrs} \rightarrow \mathbf{Crs}$ does not preserve coequalizers or colimits in general. However, it *does* preserve coproducts, and we will use these to show

that in fact **Crs** has *all* colimits (which are evidently coarsely invariant). In particular, **Crs** has all coequalizers, which contrasts with the situation in **PCrs** (recall that having all coproducts and all coequalizers would imply having all colimits).

3. THE COARSE CATEGORIES

3.1. Closeness of maps. In classical (unital) coarse geometry, two maps $f, f': Y \rightarrow X$ are *close* if $(f \times f')(1_Y)$ is an entourage of X . Closeness is an equivalence relation on maps $Y \rightarrow X$, but note that it does not involve the coarse structure on Y at all! In the nonunital case, we must modify the definition, lest closeness not even be reflexive (e.g., take $Y := X$ nonunital and $f := f' := \text{id}_X$).

Definition 3.1.1. Coarse maps $f, f': Y \rightarrow X$ are *close* (write $f \sim_{\text{cl}} f'$) if $(f \times f')(F) \in \mathcal{E}_X$ for all $F \in \mathcal{E}_Y$.

Proposition 3.1.2. *Closeness of coarse maps $Y \rightarrow X$ is an equivalence relation (on the Hom-set $\text{Hom}_{\mathbf{PCrs}}(Y, X)$).*

Proof. Reflexivity follows since coarse maps preserve entourages. Symmetry follows by taking transposes. Transitivity: Suppose $f, f', f'': Y \rightarrow X$ are coarse maps with $f \sim_{\text{cl}} f'$ and $f' \sim_{\text{cl}} f''$. For any $F \in \mathcal{E}_Y$,

$$(f \times f'')(F) \subseteq (f \times f')(1_{F \cdot Y}) \circ (f' \times f'')(F)$$

is an entourage of X since $1_{F \cdot Y} \in \mathcal{E}_Y$ (since $1_{F \cdot Y} \subseteq F \circ F^T$). \square

Like local properness, closeness is also determined “on” unital subspaces of the domain. Thus for unital coarse spaces, our notion of closeness is just the classical one.

Proposition 3.1.3. *Coarse maps $f, f': Y \rightarrow X$ are close if and only if, for every unital subspace $Y' \subseteq Y$, $f|_{Y'}$ and $f'|_{Y'}$ are close (i.e., $(f \times f')(1_{Y'}) \in \mathcal{E}_X$). Thus, for Y unital, f and f' are close if and only if $(f \times f')(1_Y) \in \mathcal{E}_X$.*

Proof. (\Rightarrow) : Immediate.

(\Leftarrow) : For $F \in \mathcal{E}_Y$, $Y' := F \cdot Y \cup Y \cdot F$ is a unital subspace of Y , and $F \in \mathcal{E}_{Y'}$. Then

$$(f \times f')(F) = (f|_{Y'} \times f'|_{Y'})(F) \in \mathcal{E}_X,$$

as required. \square

We have not used local properness at all, so we can actually define closeness for maps which preserve entourages (but are not necessarily locally proper). However, we will not need this.

The following observation is rather important.

Proposition 3.1.4. *Suppose $f, f': Y \rightarrow X$ are coarse maps. If $X = |X|_1$ has the terminal coarse structure, then f and f' are close.*

Thus if $X = |X|_1$, then for any coarse space Y there is *at most* one (but possibly no) closeness class of coarse map $Y \rightarrow X$.

Proof. This follows immediately from Proposition 1.6.14. \square

3.2. The coarse categories. Closeness, an equivalence relation on the Hom-sets of \mathbf{PCrs} , yields a quotient category

$$\mathbf{Crs} := \mathbf{PCrs} / \sim_{\text{cl}}$$

(see §2.1), which we call the *coarse category*, together with a quotient functor $\text{Quot}: \mathbf{PCrs} \rightarrow \mathbf{Crs}$. We may similarly define quotients \mathbf{CCrs} , \mathbf{UCrs} , and \mathbf{CUCrs} of \mathbf{CPCrs} , \mathbf{UPCrs} , and \mathbf{CUPCrs} , respectively. These latter categories are full subcategories of \mathbf{PCrs} , so their quotients are full subcategories of \mathbf{Crs} .

The following is clear.

Proposition 3.2.1. *Closeness is respected by composition: If $f, f': Y \rightarrow X$ and $g, g': Z \rightarrow Y$ are coarse maps with $f \sim_{\text{cl}} f'$ and $g \sim_{\text{cl}} g'$, then $f \circ g \sim_{\text{cl}} f' \circ g'$.*

This allows us to describe the arrows of \mathbf{Crs} as closeness equivalence classes of coarse maps. Denote such classes by $[f]_{\text{cl}}: Y \rightarrow X$ (or simply $[f]$ for brevity), where f is usually taken to be a representative map $Y \rightarrow X$, i.e., $\text{Quot}(f) = [f]$. However, we will use the notation $[f]: Y \rightarrow X$ for arrows $Y \rightarrow X$ in \mathbf{Crs} even when we do not have a particular f in mind.

The notion of isomorphism in \mathbf{Crs} is weaker than in \mathbf{PCrs} . A coarse map $f: Y \rightarrow X$ is a *coarse equivalence* if $[f]$ is an isomorphism in \mathbf{Crs} . In other words, f is a coarse equivalence if and only if there is a coarse map $g: X \rightarrow Y$ so that the two possible compositions are close to the identities (i.e., $[f \circ g] = [\text{id}_X]$ and $[g \circ f] = [\text{id}_Y]$).

A functor $F: \mathbf{PCrs} \rightarrow \mathcal{C}$, \mathcal{C} any category, is *coarsely invariant* if $f \sim_{\text{cl}} f'$ implies $F(f) = F(f')$. Any coarsely invariant F induces a functor $[F]: \mathbf{Crs} \rightarrow \mathcal{C}$ with $F = [F] \circ \text{Quot}$. Coarsely invariant functors send coarse equivalences to isomorphisms. For functors $F: \mathbf{PCrs} \rightarrow \mathbf{PCrs}$ (or with codomain one of the other precoarse categories), we abuse terminology and also say that F is *coarsely invariant* if $\text{Quot} \circ F: \mathbf{PCrs} \rightarrow \mathbf{Crs}$ is coarsely invariant in the previous (stronger) sense. Such a coarsely invariant functor $F: \mathbf{PCrs} \rightarrow \mathbf{PCrs}$ induces a functor $[F]: \mathbf{Crs} \rightarrow \mathbf{Crs}$; if $F: \mathbf{PCrs} \rightarrow \mathbf{CPCrs}$, then $[F]: \mathbf{Crs} \rightarrow \mathbf{CCrs}$; etc.

3.3. Crs versus CCrs. The relation between the quotient categories \mathbf{Crs} and \mathbf{CCrs} is essentially the same as that between \mathbf{PCrs} and \mathbf{CPCrs} for the following reasons, which are easy to check.

Proposition 3.3.1. *The functors $I: \mathbf{CPCrs} \hookrightarrow \mathbf{PCrs}$ and $\text{Conn}: \mathbf{PCrs} \rightarrow \mathbf{CPCrs}$ are coarsely invariant, hence induce functors $[I]: \mathbf{CCrs} \rightarrow \mathbf{Crs}$ and $[\text{Conn}]: \mathbf{Crs} \rightarrow \mathbf{CCrs}$, respectively. In fact, $[I]$ is just the inclusion and is fully faithful. Again, $[\text{Conn}] \circ [I]$ is the identity functor (now on \mathbf{CCrs}), and $[\text{Conn}]$ is left adjoint to $[I]$.*

Consequently, we get the following (exact) analogues of Corollary 2.3.4.

Corollary 3.3.2. *$[I]$ preserves limits and $[\text{Conn}]$ preserves colimits. If v is a limiting cone to (or colimiting cone from) $[I] \circ \mathcal{F}$, where $\mathcal{F}: \mathcal{J} \rightarrow \mathbf{CCrs}$, then $[\text{Conn}] \circ v$ is a limiting cone to (or colimiting cone from, respectively) $\mathcal{F} = [\text{Conn}] \circ [I] \circ \mathcal{F}$.*

Remark 3.3.3. Evidently, I and Conn “commute” with the quotient functors $\text{Quot} (\mathbf{PCrs} \rightarrow \mathbf{Crs}$ and its restriction $\mathbf{CPCrs} \rightarrow \mathbf{CCrs}$) in that

$$\text{Quot} \circ I = [I] \circ \text{Quot} \quad \text{and} \quad \text{Quot} \circ \text{Conn} = [\text{Conn}] \circ \text{Quot}.$$

The quotient functors give a map of adjunctions (see, e.g., [17, Ch. IV §7]) from (Conn, I) to $([\text{Conn}], [I])$.

Remark 3.3.4. $[I]$ is fully faithful, but $[\text{Conn}]$ is neither full nor faithful (even though Conn is faithful, though also not full): e.g., consider

$$\text{Hom}_{\mathbf{Crs}}(*, * \amalg *) \quad \text{and} \quad \text{Hom}_{\mathbf{Crs}}(\text{Conn}(|\mathbb{Z}_+|_1 \amalg |\mathbb{Z}_+|_1), |\mathbb{Z}_+|_1 \amalg |\mathbb{Z}_+|_1),$$

respectively.

3.4. \mathbf{CCrs} versus \mathbf{CCrs}^\times . After passing to the quotients by closeness, the situation with respect to *nonempty* connected coarse spaces is greatly improved. Below, we work in \mathbf{CPCrs} or its quotient \mathbf{CCrs} (or the nonempty subcategories), so all coarse spaces will be connected. Let $I: \mathbf{CPCrs}^\times \hookrightarrow \mathbf{CPCrs}$ denote the inclusion; it is coarsely invariant, hence induces $[I]: \mathbf{CCrs}^\times \hookrightarrow \mathbf{CCrs}$, which is also the inclusion and which is fully faithful. Again, the inclusion functors “commute” with the quotient functors.

Definition 3.4.1. Fix a one-point coarse space $*$. Define a functor

$$\text{AddPt}: \mathbf{CPCrs} \rightarrow \mathbf{CPCrs}^\times$$

as follows:

- (i) For a coarse space X , $\text{AddPt}(X) := X \amalg_{\mathbf{CPCrs}} *$ (coproduct in \mathbf{CPCrs}).
- (ii) For a coarse map $f: Y \rightarrow X$, $\text{AddPt}(f) := f \amalg_{\mathbf{CPCrs}} \text{id}_*$.

(It is easy to construct the functor AddPt concretely, and all functors satisfying the above are naturally equivalent.) The following are all easy to verify.

Proposition 3.4.2. $\text{AddPt}: \mathbf{CPCrs} \rightarrow \mathbf{CPCrs}^\times$ is coarsely invariant and hence induces a functor $[\text{AddPt}]: \mathbf{CCrs} \rightarrow \mathbf{CCrs}^\times$.

AddPt is not terribly useful, but $[\text{AddPt}]$ is.

Proposition 3.4.3. $[\text{AddPt}] \circ [I]$ is naturally equivalent to the identity on \mathbf{CCrs}^\times . Moreover, $[\text{AddPt}]: \mathbf{CCrs} \rightarrow \mathbf{CCrs}^\times$ is left adjoint to $[I]$.

It follows that $[\text{AddPt}]$ is naturally equivalent to a functor $[\text{AddPt}]'$ such that $[\text{AddPt}]' \circ [I]$ is equal to the identity functor. It is easy to give a natural equivalence $\text{Id}_{\mathbf{CCrs}^\times} \rightarrow [\text{AddPt}] \circ [I]$: for each (nonempty, connected) X , the canonical inclusion $\iota_X: X \rightarrow X \amalg *$ is a coarse equivalence hence an isomorphism $[\iota_X]: X \rightarrow \text{AddPt}(X)$ in \mathbf{CCrs}^\times .

Corollary 3.4.4. $[I]$ preserves limits and $[\text{AddPt}]$ preserves colimits. If v is a limiting cone to (or, a colimiting cone from) $[I] \circ \mathcal{F}$, where $\mathcal{F}: \mathcal{J} \rightarrow \mathbf{CCrs}^\times$, then $[\text{AddPt}] \circ v$ is a limiting cone to (or, respectively, a colimiting cone from) $[\text{AddPt}] \circ [I] \circ \mathcal{F}$ (or \mathcal{F} after applying a natural equivalence).

3.5. Limits in the coarse categories. We first prove our assertion that nonzero products in the nonunital coarse categories are just images (under the quotient functor) of products in the precoarse categories. We then show the nonunital coarse categories also have all equalizers of pairs of arrows. It then follows by a standard construction that the nonunital coarse categories have all nonzero limits.

Proposition 3.5.1. *Suppose $\{X_j: j \in J\}$ (J some index set) is a nonzero collection of coarse spaces. The product of the X_j in **Crs** (or in **CCrs** or **CCrs**[×], as appropriate) is just the coarse space*

$$X := \mathbf{PCrs}\text{-}\prod_{j \in J} X_j$$

(product in **PCrs**) together with the projections $[\pi_j]: X \rightarrow X_j, j \in J$ (closeness classes of the projections). Thus **Crs** (and **CCrs** and **CCrs**[×]) have all nonzero products.

Recall, from Corollary 2.4.4, that none of the quotient coarse categories has a zero product, i.e., terminal object.

Proof. The cone π in **PCrs** maps (via the quotient functor) to a cone $[\pi] := \text{Quot} \circ \pi$ in **Crs**; we must prove universality. Suppose Y is a coarse space and $[\mu_j]: Y \rightarrow X_j, j \in J$, is a collection of arrows in **Crs**. Choosing representative coarse maps $\mu_j: Y \rightarrow X_j$, we get (since the cone π is universal) a natural coarse map $t: Y \rightarrow X$ such that $\mu_j = \pi_j \circ t$ for all j . Of course, this implies $[\mu_j] = [\pi_j] \circ [t]$.

We must show that this $[t]$ is unique (hence does not depend on our choice of representatives μ_j). Suppose $[t']: Y \rightarrow X$ is a class such that $[\mu_j] = [\pi_j] \circ [t']$ for all j . Choose a representative t' . Suppose $F \in \mathcal{E}_Y$, and put $E := (t \times t')(F)$, which is in $\mathcal{E}_{|X|_1}$ by Proposition 1.6.14. For each j , we have that $\mu'_j := \pi_j \circ t' \sim_{\text{cl}} \mu_j = \pi_j \circ t$, and hence

$$(\pi_j)^{\times 2}(E) = ((\pi_j \circ t) \times (\pi_j \circ t'))(F)$$

is in \mathcal{E}_{X_j} . Moreover, since $(\mu_j \times \mu'_j)|_F = (\pi_j)^{\times 2}|_E \circ (t \times t')|_E^F$ is proper (by the same Proposition), π_j is locally proper for E . Thus $E \in (\pi_j)^* \mathcal{E}_{X_j}$ for all j , so $E \in \mathcal{E}_X$. Hence t is close to t' .

For **CCrs** and **CCrs**[×], it suffices to recall that nonzero products of connected coarse spaces are connected, and nonzero products of nonempty coarse spaces are nonempty. \square

Remark 3.5.2. For obvious reasons, we cannot usually obtain products in the unital coarse categories using the above construction. However, unlike in **PCrs**, this does not imply the nonexistence of products. In certain cases (see, e.g., Remark 3.8.11), the (nonunital) product above will be coarsely equivalent to a unital coarse space which is a product in **UCrs**. I do not know, in general, which products exist in **UCrs**.

Next, we examine equalizers in the (nonunital) coarse categories. Unlike products, equalizers in the coarse categories are not usually “the same” as equalizers in the precoarse categories.

Definition 3.5.3. Suppose X is a coarse space and $f, f': Y \rightarrow X$ are set maps (Y some set). f and f' are *pointwise connected* if $f(y)$ is connected to $f'(y)$ for all $y \in Y$. f and f' are *close* for $F \in f^* \mathcal{E}_X \cap (f')^* \mathcal{E}_X$ if $(f \times f')(F) \in \mathcal{E}_X$.

Of course, if X is connected, all set maps into X are pointwise connected. If $f: Y \rightarrow X$ is a coarse map, then any coarse map close to f is pointwise connected to f .

Lemma 3.5.4. Suppose X is a coarse space and $f, f': Y \rightarrow X$ are set maps. If f and f' are close for both $F, F' \in \mathcal{E}_{|Y|_1}$, then f and f' are close for $F + F'$, $F \circ F'$, F^Γ , and all subsets of F .

Proof. Again, the only (slightly) nontrivial one is $F \circ F'$:

$$(f \times f')(F \circ F') \subseteq f^{\times 2}(F) \circ (f \times f')(F') \in \mathcal{E}_X.$$

□

Definition 3.5.5. Suppose X is a coarse space and $f, f': Y \rightarrow X$ are pointwise connected coarse maps. The *equalizing pull-back coarse structure* $(f, f')^* \mathcal{E}_X$ (on Y along f and f') is

$$(f, f')^* \mathcal{E}_X := \{F \in f^* \mathcal{E}_X \cap (f')^* \mathcal{E}_X : f \text{ and } f' \text{ are close for } F\}.$$

Pointwise-connectedness is important: it guarantees that the singletons $\{1_y\}$, $y \in Y$, are in $(f, f')^* \mathcal{E}_X$. It then follows from the Lemma that $(f, f')^* \mathcal{E}_X$ is a coarse structure on Y .

Definition 3.5.6. Suppose $f, f': Y \rightarrow X$ are coarse maps. Define the *equalizer* of $[f]$ and $[f']$ is

$$\text{Eq}_{[f], [f']} := \{y \in Y : f(y) \text{ is connected to } f'(y)\} \subseteq Y$$

with the coarse structure

$$\mathcal{E}_{\text{Eq}_{[f], [f']}} := \mathcal{E}_Y|_{\text{Eq}_{[f], [f']}} \cap (f|_{\text{Eq}_{[f], [f']}}, f'|_{\text{Eq}_{[f], [f']}})^* \mathcal{E}_X$$

(where $\mathcal{E}_Y|_{\text{Eq}_{[f], [f']}}$ is the subspace coarse structure on $\text{Eq}_{[f], [f']} \subseteq Y$), together with closeness class of the inclusion map

$$\text{eq}_{[f], [f']} : \text{Eq}_{[f], [f']} \rightarrow Y$$

(which is coarse).

Clearly, the restrictions $f|_{\text{Eq}_{[f], [f']}}$ and $f'|_{\text{Eq}_{[f], [f']}}$ are pointwise connected, so $\mathcal{E}_{\text{Eq}_{[f], [f']}}$ really is a coarse structure. Also, the above definition does not depend on order, i.e., $\text{Eq}_{[f], [f']} = \text{Eq}_{[f'], [f]}$.

Lemma 3.5.7. Suppose $f, f': Y \rightarrow X$ are coarse maps. The equalizer of $[f]$ and $[f']$ is coarsely invariant in the sense that $\text{Eq}_{[f], [f']}$ and $[\text{eq}_{[f], [f]}]$ (indeed, $\text{eq}_{[f], [f]}$) only depend on the closeness class of f and f' (hence the notation).

Proof. Suppose $e, e': Y \rightarrow X$ are close to f, f' , respectively. Then, for all $y \in Y$, $e(y)$ is connected to $f(y)$ and $e'(y)$ is connected to $f'(y)$; for $y \in \text{Eq}_{[f], [f']} \subseteq Y$, $f(y)$ is connected to $f'(y)$ hence $e(y)$ is connected to $e'(y)$. Thus the set $\text{Eq}_{[f], [f']}$ is coarsely invariant.

It remains to show that the coarse structure $\mathcal{E}_{\text{Eq}_{[f], [f]}}$ is also coarsely invariant. Observe that

$$\mathcal{E}_{\text{Eq}_{[f], [f]}} = \{F \in \mathcal{E}_Y |_{\text{Eq}_{[f], [f]}} : (f \times f')(F) \in \mathcal{E}_X\}$$

and $\mathcal{E}_Y |_{\text{Eq}_{[f], [f]}} = \mathcal{E}_Y |_{\text{Eq}_{[f], [f]}}$. If $F \in \mathcal{E}_{\text{Eq}_{[f], [f]}}$, then

$$(e \times e')(F) \subseteq (e \times f)(1_{F \cdot Y}) \circ (f \times f')(F) \circ (f' \times e)(1_{Y \cdot F})$$

is in \mathcal{E}_X , and so $F \in \mathcal{E}_{\text{Eq}_{[e], [e]}}$; the reverse inclusion follows symmetrically. \square

Proposition 3.5.8. *The equalizer of $[f], [f'] : Y \rightarrow X$ really is (in the categorical sense) the equalizer of $[f]$ and $[f']$ in \mathbf{Crs} (or in \mathbf{CCrs} or \mathbf{CCrs}^\times , as appropriate), hence the terminology. Thus \mathbf{Crs} (and \mathbf{CCrs} and \mathbf{CCrs}^\times) have all equalizers of pairs of arrows.*

Proof. Fix representative coarse maps f and f' , and suppose $g : Z \rightarrow Y$ is a coarse map such that $f \circ g \sim_{\text{cl}} f' \circ g$. Then clearly the (set) image of g is contained in $\text{Eq}_{[f], [f']}$, and indeed

$$\tilde{g} := g|_{\text{Eq}_{[f], [f]}} : Z \rightarrow \text{Eq}_{[f], [f]}$$

is clearly coarse with $g = \text{eq}_{[f], [f]} \circ \tilde{g}$ (hence $[g] = [\text{eq}_{[f], [f]}] \circ [\tilde{g}]$).

We must prove uniqueness of $[\tilde{g}]$. Suppose $\tilde{g}' : Z \rightarrow \text{Eq}_{[f], [f]}$ is a coarse map with $g \sim_{\text{cl}} \text{eq}_{[f], [f]} \circ \tilde{g}' =: g'$. Then, for all $G \in \mathcal{E}_Z$,

$$F := (\tilde{g} \times \tilde{g}')(G) = (g \times g')(G) \in \mathcal{E}_Y$$

and, since $f \circ g' \sim_{\text{cl}} f \circ g \sim_{\text{cl}} f' \circ g \sim_{\text{cl}} f' \circ g'$, we have

$$(f \times f')(F) = ((f \circ g') \times (f' \circ g'))(F) \in \mathcal{E}_X,$$

so $F \in \mathcal{E}_{\text{Eq}_{[f], [f]}}$. Hence \tilde{g} is close to \tilde{g}' , as required.

If X and Y are connected, then $\text{Eq}_{[f], [f]}$ is clearly connected. Moreover, if X is connected, then $\text{Eq}_{[f], [f]} = Y$ as a set and hence is nonempty if Y is nonempty. \square

Remark 3.5.9. The above construction does not work in the unital coarse categories because the equalizing pull-back coarse structures are not in general unital (and one cannot “unitalize” them and still have the required properties). Again, this does not imply the nonexistence of equalizers in \mathbf{UCrs} , and I do not know which equalizers exist in \mathbf{UCrs} .

Remark 3.5.10. When X is a coarse space and $f, f' : Y \rightarrow X$ are just set maps, one can take

$$\mathcal{E}_Y := f^* \mathcal{E}_X \cap (f')^* \mathcal{E}_X$$

and apply the above Proposition. If $g : Z \rightarrow Y$ is another set map, one can then take $\mathcal{E}_Z := g^* \mathcal{E}_Y$. Then $f \circ g$ is close to $f' \circ g$ if and only if g factors through the equalizer of $[f]$ and $[f']$.

We have now shown that the nonunital coarse categories have all nonzero products and all equalizers. It follows, using a standard argument, that these categories have all nonzero limits. For completeness, we give this argument.

Theorem 3.5.11. *The nonunital coarse categories \mathbf{Crs} , \mathbf{CCrs} , and \mathbf{CCrs}^\times have all nonzero limits.*

Proof. Let \mathcal{C} be one of the above categories and suppose $\mathcal{F}_X: \mathcal{J} \rightarrow \mathcal{C}$ (\mathcal{J} nonzero, small) is a functor, putting $X_j := \mathcal{F}_X(j)$ for $j \in \text{Obj}(\mathcal{J})$ as usual. If \mathcal{J} has no arrows (i.e., $\text{Map}(\mathcal{J}) = \emptyset$), then $\mathcal{C}\text{-Lim } \mathcal{F}_X$ is just a product, and we are done.

Otherwise, let

$$Y := \mathcal{C}\text{-}\prod_{j \in \text{Obj}(\mathcal{J})} X_j$$

and

$$X := \mathcal{C}\text{-}\prod_{u \in \text{Map}(\mathcal{J})} X_{\text{target}(u)}.$$

We have two collections of arrows $[f_u], [f'_u]: Y \rightarrow X_{\text{target}(u)}$, $u \in \text{Map}(\mathcal{J})$:

$$[f_u] := [\pi_{\text{target}(u)}] \quad \text{and} \quad [f'_u] := \mathcal{F}_X(u) \circ [\pi_{\text{source}(u)}].$$

By the universal property of products, these collections of arrows give rise to canonical arrows $[f]: Y \rightarrow X$ and $[f']: Y \rightarrow X$, respectively. Put

$$\mathcal{C}\text{-Lim } \mathcal{F}_X := \text{Eq}_{[f], [f']},$$

with the cone $[nu]: \mathcal{C}\text{-Lim } \mathcal{F}_X \rightarrow \mathcal{F}_X$ being defined by $[\nu_j] := [\pi_j] \circ [\text{eq}_{[f], [f']}]$ for $j \in \text{Obj}(\mathcal{J})$. It is easy to check that $[\nu]$ is indeed a limiting cone. \square

Remark 3.5.12. It follows from the above proof that, as a set, one can always take the limit $\text{Lim } \mathcal{F}_X$ to be a subset of the product (set) $\prod_{j \in \text{Obj}(\mathcal{J})} X_j$. When all the coarse spaces X_j are connected (i.e., in \mathbf{CCrs}), one can take $\text{Lim } \mathcal{F}_X$ to be, as a set, exactly the set product. Moreover, the proof actually gives a concrete description of limits in the coarse categories. If all the X_j are connected, the coarse structure on

$$Y := \text{Lim } \mathcal{F}_X := \mathbf{Set}\text{-}\prod_{j \in \text{Obj}(\mathcal{J})} X_j$$

consists of all $F \in \mathcal{E}_{|Y|_1}$ such that, for all arrows $u \in \text{Map}(\mathcal{J})$ and (all) representative coarse maps $f_u: X_{\text{source}(u)} \rightarrow X_{\text{target}(u)}$ of $\mathcal{F}_X(u)$:

- (i) $((f_u \circ \pi_{\text{source}(u)}) \times \pi_{\text{target}(u)})|_F$ is proper; and
- (ii) $((f_u \circ \pi_{\text{source}(u)}) \times \pi_{\text{target}(u)})(F)$ is an entourage of $X_{\text{target}(u)}$.

(By taking u to be the identity arrow of $j \in \text{Obj}(\mathcal{J})$, one gets that the π_j are coarse for F .)

3.6. Entourages as subspaces of products. Is there a relation between entourages of a coarse space X , which are subsets of $X^{\times 2} := X \times X$, and the product coarse space $X \times X$? We first need a coarse space $\text{Term}(X)$ which we will discuss more thoroughly in §3.8: For any X , $\text{Term}(X) := X$ as a set, with coarse structure

$$\mathcal{E}_{\text{Term}(X)} := \{E \in \mathcal{E}_{|X|_1} : 1_{E \cdot X}, 1_{X \cdot E} \in \mathcal{E}_X\}.$$

Note that if X is unital, $\text{Term}(X) = |X|_1$.

The following will be useful later in conjunction with various universal properties, as well as generalized coarse quotients (which we intend to study in [15]).

Proposition 3.6.1. *Suppose X is a coarse space. If $E \in \mathcal{E}_{\text{Term}(X)}$, then E can be considered as a unital subspace $|E|$ of the product coarse space $X \times X$. If in fact $E \in \mathcal{E}_X$, then the restricted projections $\pi_1|_{|E|}, \pi_2|_{|E|} : |E| \rightarrow X$ are close. Conversely, any unital subspace $|E| \subseteq X \times X$ determines a subset $E \in \mathcal{E}_{\text{Term}(X)} \subseteq \mathcal{E}_{|X|_1}$; if $\pi_1|_{|E|}, \pi_2|_{|E|}$ are close, then $E \in \mathcal{E}_X$.*

Proof. If $E \in \mathcal{E}_{\text{Term}(X)}$, then $1_{|E|}$ is an entourage of $X \times X$: certainly $1_{|E|} \in \mathcal{E}_{|X \times X|_1}$, and $(\pi_1 \times \pi_1)(1_{|E|}) = 1_{E \cdot X}$ and $(\pi_2 \times \pi_2)(1_{|E|}) = 1_{X \cdot E}$ are entourages of X . If $E \in \mathcal{E}_X$, then $(\pi_1 \times \pi_2)(1_{|E|}) = E \in \mathcal{E}_X$; since $|E|$ is unital, it follows that the restricted projections are close.

Conversely, suppose $|E| \subseteq X \times X$ is a unital subspace. Then the restricted projections $\pi_1|_E = \pi_1|_{|E|}$ and $\pi_2|_E = \pi_2|_{|E|}$ are proper, so $E \in \mathcal{E}_{|X|_1}$. Since π_1 maps unital subspaces of $X \times X$ to unital subspaces of X and $\pi_1(|E|) = E \cdot X$, the left support, and symmetrically the right support, of E is a unital subspace of X , and so $E \in \mathcal{E}_{\text{Term}(X)}$. If the restricted projections are close, then $E = (\pi_1 \times \pi_2)(1_{|E|}) \in \mathcal{E}_X$. \square

3.7. Colimits in the coarse categories. We now do the same for coproducts, coequalizers, and thus colimits in the coarse categories.

Proposition 3.7.1. *Suppose that \mathcal{C} is one of the coarse categories **Crs**, **CCrs**, **UCrs**, or **CUCrs**, that \mathcal{PC} is the corresponding precoarse category, and that $\{Y_j : j \in J\}$ (J some index set) is a collection of coarse spaces in \mathcal{C} (or \mathcal{PC}). The coproduct of the Y_j in \mathcal{C} is just the coarse space*

$$Y := \mathcal{PC}\text{-}\coprod_{j \in J} Y_j$$

(coproduct in \mathcal{PC}) together with the “inclusions” $[\iota_j] : Y_j \rightarrow Y, j \in J$ (closeness classes of the inclusions). If instead $\mathcal{C} = \mathbf{CCrs}^\times$, then the same holds except when $J = \emptyset$, in which case the coproduct is any one-point coarse space. Thus all the coarse categories have all coproducts.

Proof. We have shown (or at least mentioned, in the unital cases) the existence of the corresponding coproduct cone ι in the corresponding precoarse category, leaving aside the special case of $\mathcal{C} = \mathbf{CCrs}^\times$ and $J = \emptyset$ (which is easily handled). The quotient functor yields a cone $[\iota]$ in the coarse category \mathcal{C} ; we must show that it is universal.

Suppose X is a coarse space and $[\mu_j]: Y_j \rightarrow X, j \in J$, is a collection of arrows in \mathcal{C} . Choosing representative coarse maps μ_j , we get a natural coarse map $t: Y \rightarrow X$ such that $\mu_j = t \circ \iota_j$ (and hence $[\mu_j] = [\pi_j \circ [t]]$ for all j). We must show this $[t]$ is unique. Suppose $t': Y \rightarrow X$ is such that $\mu_j \sim_{\text{cl}} t' \circ \iota_j$ for all j . The coarse structure on the precoarse coproduct Y is generated by $F := (\iota_j)^{\times 2}(F_j), F_j \in \mathcal{E}_{Y_j}, j \in J$, and so to show $t \sim_{\text{cl}} t'$ it is enough to show that $(t \times t')(F) \in \mathcal{E}_X$ for such F . But

$$(t \times t')(F) = ((t \circ \iota_j) \times (t' \circ \iota_j))(F_j)$$

is in \mathcal{E}_X since $t \circ \iota_j = \mu_j \sim_{\text{cl}} t' \circ \iota_j$, as required. \square

Next, coequalizers: Unlike coproducts, coequalizers in the coarse categories differ from coequalizers in the precoarse categories; in particular, they always exist.

Definition 3.7.2. Suppose Y is a coarse space and $f, f': Y \rightarrow X$ (X some set) are locally proper maps. The *coequalizing push-forward coarse structure* $(f, f')_* \mathcal{E}_Y$ (on X along f and f') is

$$(f, f')_* \mathcal{E}_Y := \langle f_* \mathcal{E}_Y, (f')_* \mathcal{E}_Y, \{(f \times f')(F): F \in \mathcal{E}_Y\} \rangle_Y.$$

(We may similarly define connected, unital, and connected unital versions.)

By Proposition 1.6.14, the sets $(f \times f')(F)$ satisfy the properness axiom. The coequalizing push-forward coarse structure makes f and f' *close* coarse maps, and is the minimum coarse structure on X for which this is true.

Definition 3.7.3. Suppose $f, f': Y \rightarrow X$ are coarse maps. The *coequalizer* of $[f]$ and $[f']$ is $\text{Coeq}_{[f], [f']} := X$ equipped the coarse structure

$$\mathcal{E}_{\text{Coeq}_{[f], [f']}} := \langle \mathcal{E}_X, (f, f')_* \mathcal{E}_Y \rangle_X,$$

together with the closeness class of “identity” map

$$\text{coeq}_{[f], [f']}: X \rightarrow \text{Coeq}_{[f], [f']}$$

(which is a coarse map).

Observe that if X is unital so too is the coequalizer, and similarly if X is connected.

Lemma 3.7.4. Suppose $f, f': Y \rightarrow X$ are coarse maps. The coequalizer of $[f]$ and $[f']$ is coarsely invariant (hence the notation).

Proof. Suppose $e, e': Y \rightarrow X$ are close to f, f' , respectively. Observe that, since $f_* \mathcal{E}_Y, (f')_* \mathcal{E}_Y \subseteq \mathcal{E}_X$,

$$\mathcal{E}_{\text{Coeq}_{[f], [f']}} = \langle \mathcal{E}_X, \{(f \times f')(F): F \in \mathcal{E}_Y\} \rangle_X$$

and similarly for e and e' . Thus it suffices to show

$$\{(e \times e')(F): F \in \mathcal{E}_Y\} \subseteq \mathcal{E}_{\text{Coeq}_{[f], [f']}}$$

and similarly symmetrically. But if $F \in \mathcal{E}_Y$, then

$$(e \times e')(F) \subseteq (e \times f)(1_{F.Y}) \circ (f \times f')(F) \circ (f' \times e')(1_{Y.F})$$

is in $\mathcal{E}_{\text{Coeq}_{[f],[f']}}$, as required. \square

Proposition 3.7.5. *The coequalizer of $[f], [f'] : Y \rightarrow X$ really is (in the categorical sense) the coequalizer of $[f]$ and $[f']$ in \mathbf{Crs} (or in \mathbf{CCrs} , \mathbf{CCrs}^\times , \mathbf{UCrs} , or \mathbf{CUCrs} , as appropriate), hence the terminology. Thus \mathbf{Crs} (and the other coarse categories) have all coequalizers of pairs of arrows.*

Proof. Fix representative coarse maps f and f' , and suppose $g : X \rightarrow W$ is a coarse map such that $g \circ f \sim_{\text{cl}} g \circ f'$. Let $\underline{g} : \text{Coeq}_{[f],[f']} \rightarrow W$ be the same, as a set map, as g ; then clearly $\underline{g} = g \circ \text{coeq}_{[f],[f']}$, and hence $[\underline{g}] = [g] \circ [\text{coeq}_{[f],[f]}]$, assuming \underline{g} is actually a coarse map. To show that \underline{g} is coarse, it suffices to show that \underline{g} is coarse for sets $E := (f \times f')(F)$, $F \in \mathcal{E}_Y$. Since

$$((g \circ f) \times (g \circ f'))|_F = g^{\times 2}|_E \circ (f \times f')|_F^E$$

is proper (Proposition 1.6.14), it follows that $\underline{g}^{\times 2}|_E = g^{\times 2}|_E$ is proper, hence \underline{g} is locally proper for E . Since $g \circ f$ and $g \circ f'$ are close, it follows that \underline{g} preserves E .

Uniqueness of $[\underline{g}]$: Suppose $\underline{g}' : \text{Coeq}_{[f],[f']} \rightarrow W$ is a coarse map such that $\underline{g} \sim_{\text{cl}} \underline{g}' \circ \text{coeq}_{[f],[f']}$. To show that \underline{g} is close to \underline{g}' , we must show that $(\underline{g} \times \underline{g}')(E) \in \mathcal{E}_W$ for all $E \in \mathcal{E}_{\text{Coeq}_{[f],[f]}}$. Clearly, this is the case for $E \in \mathcal{E}_X \subseteq \mathcal{E}_{\text{Coeq}_{[f],[f]}}$, so it suffices to show this for $E = (f \times f')(F)$ for some $F \in \mathcal{E}_Y$. The map $\underline{g}' := \underline{g}' \circ \text{coeq}_{[f],[f']}$ is close to \underline{g} , hence $\underline{g} \circ f \sim_{\text{cl}} \underline{g}' \circ f'$. Therefore,

$$(\underline{g} \times \underline{g}')((f \times f')(F)) = ((\underline{g} \circ f) \times (\underline{g}' \circ f'))(F),$$

is in \mathcal{E}_W , as required.

As previously noted, if X is connected, unital, and/or nonempty, then $\text{Eq}_{[f],[f']}$ has the corresponding property or properties, so the above actually proves the result in all the coarse categories. \square

Since the coarse categories have all coproducts and coequalizers, we immediately get the following.

Theorem 3.7.6. *The coarse categories \mathbf{Crs} , \mathbf{CCrs} , \mathbf{CCrs}^\times , \mathbf{UCrs} , and \mathbf{CUCrs} have all colimits.*

3.8. The termination functor. For essentially set theoretic reasons, \mathbf{Crs} does not have a terminal object (Corollary 2.4.4). However, for many purposes, one can find a suitable substitute. We begin with some general definitions which are applicable in any category \mathcal{C} .

Definition 3.8.1. In \mathcal{C} , an object \tilde{X} *terminates* an object X if:

- (i) there is a (unique) arrow $\tau_X : X \rightarrow \tilde{X}$; and
- (ii) for all $Y \in \text{Obj}(\mathcal{C})$, there is at most one arrow $Y \rightarrow \tilde{X}$.

I.e., \tilde{X} is terminal in the full subcategory of \mathcal{C} consisting of X and all objects mapping to \tilde{X} . \tilde{X} *universally terminates* X if it is the smallest object terminating X (i.e., for all \tilde{X}' terminating X there is an arrow $\tilde{X} \rightarrow \tilde{X}'$).

If \tilde{X} terminates X , then for all Y and pairs of arrows $f, g: Y \rightarrow X$, $\tau_X \circ f = \tau_X \circ g$. Two objects universally terminating X are canonically and uniquely isomorphic. If \tilde{X} terminates any object, then it universally terminates itself.

In a category with a terminal object 1 , the product of any object Y and 1 is just Y . The following generalizes this.

Proposition 3.8.2. *If there is some arrow $f: Y \rightarrow X$ in \mathcal{C} and \tilde{X} terminates X in \mathcal{C} , then Y is the (categorical) product of \tilde{X} and Y (in \mathcal{C}).*

Proof. The two “projections” from Y are $\pi_{\tilde{X}} := \tau_X \circ f: Y \rightarrow \tilde{X}$ and $\pi_Y := \text{id}_Y: Y \rightarrow Y$. Suppose $Z \in \text{Obj}(\mathcal{C})$ is equipped with arrows $p_{\tilde{X}}: Z \rightarrow \tilde{X}$ and $p_Y: Z \rightarrow Y$. Both these arrows factor through p_Y : evidently $p_Y = \pi_Y \circ p_Y$, but also $p_{\tilde{X}} = \pi_{\tilde{X}} \circ p_Y$ since there is only one arrow $Z \rightarrow \tilde{X}$. \square

If \mathcal{C} is known to have products (of pairs of objects), we can restate the above Proposition in the following way: Whenever there is an arrow $f: Y \rightarrow X$ and \tilde{X} terminates X , the projection $\pi_Y: \tilde{X} \times Y \rightarrow Y$ is an isomorphism. Moreover, one the inverse isomorphism is given by the composition

$$Y \xrightarrow{\Delta_Y} Y \times Y \xrightarrow{(\tau_X \circ f) \times \text{id}_Y} \tilde{X} \times Y.$$

Definition 3.8.3. A *termination functor* on \mathcal{C} is a functor $\mathcal{C} \rightarrow \mathcal{C}$ (temporarily denoted $X \mapsto \tilde{X}$) which sends each X to an object \tilde{X} terminating X ; such a functor is *universal* if \tilde{X} always universally terminates X .

The following is implied: Whenever there is an arrow $f: Y \rightarrow X$, there is a unique arrow $\tilde{Y} \rightarrow \tilde{X}$ (namely \tilde{f}). Note that universality is meant in the “point-wise” sense, and we do not assert universality as a termination functor. Universal termination functors are unique up to natural equivalence. Also observe that universal termination functors are idempotent up to natural equivalence.

Example 3.8.4. If \mathcal{C} has a terminal object 1 , then 1 terminates all objects, and $X \mapsto 1$ is a termination functor (not necessarily universal). In \mathbf{Set}_* or \mathbf{Top}_* (pointed sets or topological spaces, respectively), the functor $X \mapsto *$, where $*$ is any one-point set or space, is a universal termination functor. More generally, in any category \mathcal{C} with a zero object 0 (i.e., 0 is initial and terminal), $X \mapsto 0$ is a universal termination functor.

Example 3.8.5. In \mathbf{Set} or \mathbf{Top} , the functor given by

$$X \mapsto \begin{cases} \emptyset & \text{if } X = \emptyset, \text{ or} \\ * & \text{if } X \neq \emptyset, \end{cases}$$

is a universal termination functor.

Example 3.8.6. In \mathbf{Crs} (and our various full subcategories), $|X|_1$ terminates any coarse space X (Proposition 3.1.4). However, $X \mapsto |X|_1$ does not define a functor on \mathbf{PCrs} (or \mathbf{Crs}). E.g., for any set X , there is always a (unique) coarse map from $|X|_0$ to a one-point coarse space $*$, but no coarse map $|X|_1 = ||X|_0|_1 \rightarrow *$ when X is infinite. The problem is that coarse maps from $|X|_1$ must be globally proper; in the unital categories this is not a problem, so $X \mapsto |X|_1$ does define a coarsely

invariant functor $\mathbf{UPCrS} \rightarrow \mathbf{UPCrS}$ (for example). The induced functor on unital coarse category \mathbf{UCrS} is a universal termination functor. We wish to generalize this to all of \mathbf{CrS} .

We recall the definition of the coarse space $\text{Term}(X)$ (for X a coarse space) from §3.6, and extend Term to a functor in the obvious way.

Definition 3.8.7. For any coarse space X , $\text{Term}(X)$ is the coarse space which is just X as a set with coarse structure

$$\mathcal{E}_{\text{Term}(X)} := \{E \in \mathcal{E}_{|X|_1} : 1_{E \cdot X}, 1_{X \cdot E} \in \mathcal{E}_X\};$$

$\tau_X: X \rightarrow \text{Term}(X)$ is the “identity” map. If $f: Y \rightarrow X$, $\text{Term}(f): \text{Term}(Y) \rightarrow \text{Term}(X)$ is the same as f as a set map.

Observe the following:

- (i) $E \subseteq X^{\times 2}$ is an entourage of $\text{Term}(X)$ if and only if E satisfies the properness axiom (i.e., $E \in \mathcal{E}_{|X|_1}$) and the left and right supports of E are unital subspaces of X .
- (ii) $\text{Term}(X)$ has the same unital subspaces as X and is the maximum coarse structure on X with this property. (Consequently, if X is unital, $\text{Term}(X) = |X|_1$. It also follows that Term is idempotent, and hence so too is the induced functor $[\text{Term}]$; see below.)

Proposition 3.8.8. *Term is a coarsely invariant functor $\mathbf{PCrS} \rightarrow \mathbf{PCrS}$. The induced functor $[\text{Term}]: \mathbf{CrS} \rightarrow \mathbf{CrS}$ is a universal termination functor.*

Proof. That $\text{Term}(f)$ is a coarse map follows from the above observations, and hence $\text{Term}(f)$ is a functor. Moreover, using the above observations, we see that, for all X , all coarse maps to $\text{Term}(X)$ are close. In particular, this implies first that Term is coarsely invariant and second that $[\text{Term}]$ is a termination functor on \mathbf{CrS} .

It only remains to show universality. Suppose \tilde{X} terminates X , so there is a unique $[t]: X \rightarrow \tilde{X}$, represented by a coarse map t , say. It suffices to show that there is a coarse map $t': \text{Term}(X) \rightarrow \tilde{X}$; since \tilde{X} terminates X in \mathbf{CrS} , uniqueness of $[t']$ follows, as does the equality $[t] = [t'] \circ [\tau_X]$.

Take $t' := t: \text{Term}(X) = X \rightarrow \tilde{X}$ as a set map. Local properness of t' follows from the above observations and Proposition 1.6.1(ii). To see that t' preserves entourages, we use Proposition 3.6.1: If $E \in \mathcal{E}_{\text{Term}(X)}$, consider the unital subspace $|E|$ of the product coarse space $X \times X$. Since \tilde{X} terminates X , $t \circ \pi_1|_{|E|} \sim_{\text{cl}} t \circ \pi_2|_{|E|}$, and hence

$$((t \circ \pi_1|_{|E|}) \times (t \circ \pi_2|_{|E|}))(1_{|E|}) = (t')^{\times 2}(E)$$

is an entourage of \tilde{X} , as required. \square

In the above proof, one could instead consider the map $\text{Term}(t): \text{Term}(X) \rightarrow \text{Term}(\tilde{X})$, and show that $\text{Term}(\tilde{X}) = \tilde{X}$.

Remark 3.8.9. Term restricts to (coarsely invariant) endofunctors on the other precoarse categories, and hence $[\text{Term}]$ restricts to universal termination functors on the other coarse categories. (The proof of the above Proposition requires only unital coarse spaces $|E|$ and not actually the nonunital products $X \times X$, and hence works even in the unital cases.) Of course, in the unital cases, Term is just the functor $X \mapsto |X|_1$.

By applying Proposition 3.8.2, we immediately get the following, which will play a crucial role in the development of exponential objects in the coarse categories [13].

Corollary 3.8.10. *If there is a coarse map $Y \rightarrow \text{Term}(X)$, where X and Y are coarse spaces, then*

$$\pi_Y: \text{Term}(X) \times Y \rightarrow Y$$

is a coarse equivalence. The maps

$$D_\tau := (\tau \times \text{id}_Y) \circ \Delta_Y: Y \rightarrow \text{Term}(X) \times Y,$$

*where $\tau: Y \rightarrow \text{Term}(X)$ is any coarse map (they are all close), are coarsely inverse to π_Y . Hence, if there is a coarse map $Y \rightarrow \text{Term}(X)$, then $Y \cong \text{Term}(X) \times Y$ canonically in **Crs** (or in **CCrs** or **CCrs**[×]). In the case $Y := X$, we get that $\pi_X: \text{Term}(X) \times X \rightarrow X$ and $D_X := D_{\tau_X}: X \rightarrow \text{Term}(X) \times X$ are coarsely inverse coarse equivalences, so $X \cong \text{Term}(X) \times X$ canonically in **Crs** (or in **CCrs** or **CCrs**[×]).*

Remark 3.8.11. For any set X , $\text{Term}(|X|_1) = |X|_1$, so $|X|_1 \times |X|_1$ is (canonically) coarsely equivalent to $|X|_1$. While $|X|_1$ is always unital, $|X|_1 \times |X|_1$ is unital only when X is finite. In particular, unitality is *not* coarsely invariant. It also follows easily that $|X|_1$ is actually the product of $|X|_1$ with itself in the unital coarse category **UCrs**. More generally, for any coarse space X , the product of X and $|X|_1$ in **UCrs** is just X . (As previously mentioned, **UCrs** has some products of infinite spaces, even though the natural construction of the corresponding products in **Crs** are nonunital.)

3.9. Monics and images.

Example 3.9.1. Pull-back coarse structures are not coarsely invariant. That is, suppose $f, f': Y \rightarrow X$ are coarse maps. Even if $f \sim_{\text{cl}} f'$, it may not be the case that $f^* \mathcal{E}_X = (f')^* \mathcal{E}_X$. To see this, take $Y := |\mathbb{N}|_0^{\text{C}}$, $X := |\mathbb{N}|_1$, f to be the “identity” map (as a set map), and f' to be a constant map. Then $f^* \mathcal{E}_X = \mathcal{E}_{|Y|_1}$ whereas $(f')^* \mathcal{E}_X = \mathcal{E}_Y$.

Proposition 3.9.2. *If $f, f': Y \rightarrow X$ are coarse maps with $f \sim_{\text{cl}} f'$, then*

$$\mathcal{E}_{\text{Term}(Y)} \cap f^* \mathcal{E}_X = \mathcal{E}_{\text{Term}(Y)} \cap (f')^* \mathcal{E}_X.$$

Proof. We prove inclusion \subseteq ; containment \supseteq follows symmetrically. Suppose $F \in \mathcal{E}_{\text{Term}(Y)} \cap f^* \mathcal{E}_X$. Since $F \in \text{Term}(Y)$, f' is locally proper for F (Proposition 1.6.1(ii)). It only remains to show that $(f')^{\times 2}(F) \in \mathcal{E}_X$. But

$$(f')^{\times 2}(F) \subseteq (f' \times f)(1_{F \cdot Y}) \circ f^{\times 2}(F) \circ (f \times f')(1_{Y \cdot F}) \in \mathcal{E}_X$$

since $f \sim_{\text{cl}} f'$ (and the left and right supports of F are unital subspaces of Y) and $f^{\times 2}(F) \in \mathcal{E}_X$. \square

Definition 3.9.3. Suppose $[f]: Y \rightarrow X$. The *coarsely invariant pull-back coarse structure* $[f]^*\mathcal{E}_X$ on Y (along $[f]$) is given by

$$[f]^*\mathcal{E}_X := \mathcal{E}_{\text{Term}(Y)} \cap f^*\mathcal{E}_X$$

(where $f: Y \rightarrow X$ is any representative coarse map).

Proposition 3.9.4. If $[f]: Y \rightarrow X$ is represented by a coarse map f , then $[f]$ factors as

$$Y \xrightarrow{[\beta]} |Y|_{[f]^*\mathcal{E}_X} \xrightarrow{[f]} X,$$

where $\beta = \text{id}_Y$ and $f = f$ as set maps (i.e., $\mathcal{E}_Y \subseteq [f]^*\mathcal{E}_X$). Moreover, $[f]$ depends only on $[f]$ (and not on the particular f) and is unique in the above factorization.

Proof. The factorization follows immediately from Corollary 1.7.3. We now show that $f \sim_{\text{cl}} f'$ implies $\tilde{f} \sim_{\text{cl}} \tilde{f}'$ (noting that $[f]^*\mathcal{E}_X = [f']^*\mathcal{E}_X$). If $F \in [f]^*\mathcal{E}_X$, then

$$\begin{aligned} (f \times f')(F) &= (f \times f')(F \circ 1_{Y \cdot F}) \\ &\subseteq f^{\times 2}(F) \circ (f \times f')(1_{Y \cdot F}) \end{aligned}$$

is in \mathcal{E}_X since $f^{\times 2}(F) \in \mathcal{E}_X$ and $1_{Y \cdot F} \in \mathcal{E}_Y$ so $(f \times f')(1_{Y \cdot F}) \in \mathcal{E}_X$ as $f \sim_{\text{cl}} f'$. Uniqueness: If $[f] = [g] \circ [\beta]$, where $[g]: |Y|_{[f]^*\mathcal{E}_X} \rightarrow X$ and g is any representative, then $f \sim_{\text{cl}} g \circ \beta$, so $\tilde{f} \sim_{\text{cl}} (g \circ \beta)_{\sim} = g$. \square

Proposition 3.9.5. $[f]: Y \rightarrow X$ is monic in **Crs** if and only if $\mathcal{E}_Y = [f]^*\mathcal{E}_X$ (i.e., if and only if $Y = |Y|_{[f]^*\mathcal{E}_X}$).

Proof. Fix a representative coarse map $f: Y \rightarrow X$ and let $Y \xrightarrow{\beta} |Y|_{[f]^*\mathcal{E}_X} \xrightarrow{[f]} X$ be the canonical factorization.

(\Rightarrow): Suppose there exists some $F \in [f]^*\mathcal{E}_X \setminus \mathcal{E}_Y$. Consider $|F|$ as a unital subspace of the product $Y \times Y$, with projections $\pi_1|_{|F|}, \pi_2|_{|F|}: |F| \rightarrow Y$. Then

$$(\pi_1|_{|F|} \times \pi_2|_{|F|})(1_{|F|}) = F,$$

so $\pi_1|_{|F|}$ is not close to $\pi_2|_{|F|}$, but $\beta \circ \pi_1|_{|F|}$ is close to $\beta \circ \pi_2|_{|F|}$. Hence $[\pi_1|_{|F|}] \neq [\pi_2|_{|F|}]$ but

$$[f] \circ [\pi_1|_{|F|}] = [f] \circ [\beta] \circ [\pi_1|_{|F|}] = [f] \circ [\beta] \circ [\pi_2|_{|F|}] = [f] \circ [\pi_2|_{|F|}],$$

so $[f]$ is not monic.

(\Leftarrow): Suppose $g, g': Z \rightarrow Y$ are coarse maps such that $[f] \circ [g] = [f] \circ [g']$. Then, for each $G \in \mathcal{E}_Z$,

$$((f \circ g) \times (f \circ g'))(G) = f^{\times 2}((g \times g')(G)) \in \mathcal{E}_X.$$

But then $(g \times g')(G) \in [f]^*\mathcal{E}_X = \mathcal{E}_Y$, so $[g] = [g']$, as required. \square

Corollary 3.9.6. For any $[f]: Y \rightarrow X$, the canonical arrow

$$[f]: |Y|_{[f]^*\mathcal{E}_X} \rightarrow X$$

is monic in **Crs**.

Definition 3.9.7. Suppose $[f]: Y \rightarrow X$. Denote $\text{Im}[f] := |Y|_{[f]^*\mathcal{E}_X}$ and $\text{im}[f] := [f]: \text{Im}[f] \rightarrow X$, where $[f]$ is defined as above. We will also sometimes write $\tilde{[f]}(Y) := \text{Im}[f]$.

Despite the notation, $[f](Y)$ should not be considered as a subspace of X (however, see Proposition 3.11.3 and the discussion which precedes it).

Theorem 3.9.8. For any $[f]: Y \rightarrow X$, the subobject of X represented by the arrow $\text{im}[f]: \text{Im}[f] \rightarrow X$ is the (categorical) image of $[f]$ in **Crs**.

Proof. Suppose $[f]$ also factors as $Y \xrightarrow{[h]} Z \xrightarrow{[g]} X$ where $[g]$ is monic, so that $\mathcal{E}_Z = [g]^*\mathcal{E}_X$. We must show that there is a unique $[h]: \text{Im}[f] \rightarrow Z$ such that $\text{im}[f] = [g] \circ [h]$.

Pick a representative coarse map $h: Y \rightarrow Z$, and put $\underline{h} := h$ as a set map $\text{Im}[f] = Y \rightarrow Z$. First, \underline{h} is a coarse map: Local properness is equivalent to properness when restricted to unital subspaces (Corollary 1.6.5); since $\text{Im}[f]$ and Y have the same unital subspaces (and $\underline{h} = h$ as set maps), \underline{h} is locally proper. Reasoning similarly, for any $F \in \mathcal{E}_{\text{Im}[f]} = [f]^*\mathcal{E}_X$, $\underline{h}^{\times 2}(F)$ is in $\mathcal{E}_{\text{Term}(Z)}$. Then, since $\mathcal{E}_Z = [g]^*\mathcal{E}_X$, it follows that \underline{h} is coarse. From the uniqueness assertion of Proposition 3.9.4, we get that $[g] \circ [\underline{h}] = \text{im}[f]$. Uniqueness of $[\underline{h}]$: If $[h']: \text{Im}[f] \rightarrow Z$ and $[g] \circ [h'] = \text{im}[f] = [g] \circ [\underline{h}]$, then $[h] = [h']$ since $[g]$ is monic. \square

3.10. Epis and coimages. For rather trivial reasons, push-forward coarse structures are not coarsely invariant. Recall that coarse structures are semirings, which gives rise to an obvious notion of ideals.

Definition 3.10.1. Suppose \mathcal{E}_X is a coarse structure on a set X . A subset $\mathcal{E} \subseteq \mathcal{E}_X$ is an *ideal* of \mathcal{E}_X if it is a coarse structure on X such that $E \circ E', E' \circ E \in \mathcal{E}$ for all $E \in \mathcal{E}, E' \in \mathcal{E}_X$. Note that any intersection of ideals is again an ideal. The *ideal* $\langle\langle \mathcal{E} \rangle\rangle_X$ (of \mathcal{E}_X generated by \mathcal{E}) is the smallest ideal of \mathcal{E}_X which contains \mathcal{E} .

Proposition 3.10.2. Suppose $f, f': Y \rightarrow X$ are coarse maps with $f \sim_{\text{cl}} f'$. Then

$$\langle\langle f_*\mathcal{E}_Y \rangle\rangle_X = \langle\langle (f')_*\mathcal{E}_Y \rangle\rangle_X.$$

Proof. Elements $E \in \langle\langle f_*\mathcal{E}_Y \rangle\rangle_X$ are exactly subsets

$$E \subseteq E' \circ f^{\times 2}(F) \circ E'' \cup E'''$$

for some $F \in \mathcal{E}_Y$ and some $E', E'', E''' \in \mathcal{E}_X$ with E''' finite. But then

$$E \subseteq (E' \circ (f \times f')(1_{F,Y})) \circ (f')^{\times 2}(F) \circ ((f' \times f)(1_{Y,F}) \circ E'') \cup E'''$$

is in $\langle\langle (f')_*\mathcal{E}_Y \rangle\rangle_X$ (and symmetrically) as required. \square

Definition 3.10.3. Suppose $[f]: Y \rightarrow X$. The *coarsely invariant push-forward coarse structure* $[f]_*\mathcal{E}_Y$ on X (along $[f]$) is given by

$$[f]_*\mathcal{E}_Y := \langle\langle f_*\mathcal{E}_Y \rangle\rangle_X$$

(where $f: Y \rightarrow X$ is any representative coarse map).

Despite the obvious parallels with coarsely invariant pull-backs, the coarsely invariant push-forward $[f]_*\mathcal{E}_Y$ depends very little on \mathcal{E}_Y . In fact, it depends only on the set of unital subspaces of Y (recall from Proposition 3.1.3 that closeness is entirely determined on the unital subspaces). Thus we have the following.

Proposition 3.10.4. *For any $[f]: Y \rightarrow X$,*

$$[f]_*\mathcal{E}_Y = (\text{im}[f])_*\mathcal{E}_{\text{Im}[f]}.$$

Proof. Recall that $\text{Im}[f] := |Y|_{[f]^*\mathcal{E}_X}$, where $[f]^*\mathcal{E}_X := \mathcal{E}_{\text{Term}(Y)} \cap f^*\mathcal{E}_X$ (for any representative map f) and $\text{im}[f] := f$ as a set map. Since $\mathcal{E}_Y \subseteq [f]^*\mathcal{E}_X$, $[f]_*\mathcal{E}_Y \subseteq (\text{im}[f])_*\mathcal{E}_{\text{Im}[f]}$. For the opposite inclusion, it suffices to show that, for $F \in [f]^*\mathcal{E}_X$,

$$E := f^{\times 2}(F) \in \langle f_*\mathcal{E}_Y \rangle_X;$$

but $F \cdot Y$ is a unital subspace of Y (hence $1_{F \cdot Y} \in \mathcal{E}_Y$) and $f^{\times 2}(F) \in \mathcal{E}_X$, so

$$E = f^{\times 2}(1_{F \cdot Y}) \circ E \in \langle f_*\mathcal{E}_Y \rangle_X,$$

as required. \square

Suppose $[f]: Y \rightarrow X$, represented by a coarse map f . Denote

$$X_{[f]} := \{x \in X: x \text{ is connected to some } x' \in f(Y)\} \subseteq X,$$

a subspace of X . It is easy to see that $X_{[f]}$ really only depends on the closeness class $[f]$, as the notation indicates. (If X is connected, then of course $X_{[f]} = X$.)

The subspace $X_{[f]} \subseteq X$ contains the set image of f (and indeed of any coarse map close to f), and hence we may take the range restriction $f|^{X_{[f]}}$ which is evidently a coarse map $Y \rightarrow X_{[f]}$. It is easy to see that the closeness class $[f|^{X_{[f]}}]$ only depends on the closeness class $[f]$, and hence we also temporarily denote

$$[f]|^{X_{[f]}} := [f|^{X_{[f]}}]: Y \rightarrow X_{[f]}.$$

Now, we may coarsely invariantly push \mathcal{E}_Y forward along $[f]|^{X_{[f]}}$ to get a coarse space $|X_{[f]}|_{([f]|^{X_{[f]}})_*\mathcal{E}_Y}$. We get the following.

Proposition 3.10.5. *If $[f]: Y \rightarrow X$ is represented by a coarse map f , then $[f]$ factors as*

$$Y \xrightarrow{[\tilde{f}]} |X_{[f]}|_{([f]|^{X_{[f]}})_*\mathcal{E}_Y} \xrightarrow{[\alpha]} X,$$

where $\tilde{f} = f|^{X_{[f]}}$ and α is the inclusion as set maps (thus $([f]|^{X_{[f]}})_*\mathcal{E}_Y \subseteq \mathcal{E}_X$). Moreover, $[\tilde{f}]$ depends only on $[f]$ (and not f) and is unique in the above factorization.

Proof. Nearly all the assertions are clear from the definitions, Corollary 1.7.6, and the previous remarks. We show that $f \sim_{\text{cl}} f'$ implies $\tilde{f} \sim_{\text{cl}} \tilde{f}'$: If $F \in \mathcal{E}_Y$, then

$$(\tilde{f} \times \tilde{f}')(F) = (f \times f')(F) \subseteq f^{\times 2}(F) \circ (f \times f')(1_{Y \cdot F})$$

is in $([f]|^{X_{[f]}})_*\mathcal{E}_Y$ since $f^{\times 2}(F) \in (f|^{X_{[f]}})_*\mathcal{E}_Y$ and $(f \times f')(1_{Y \cdot F}) \in \mathcal{E}_X|_{X_{[f]}}$.

Uniqueness: If $[f] = [\alpha] \circ [g]$, where $[g]: Y \rightarrow |X_{[f]}|_{([f]|^{X_{[f]}})_*\mathcal{E}_Y}$ and g is any representative, then $f \sim_{\text{cl}} \alpha \circ g$, so $\tilde{f} \sim_{\text{cl}} (\alpha \circ g) = g$. \square

Proposition 3.10.6. $[f]: Y \rightarrow X$ is epi in **Crs** if and only if $X_{[f]} = X$ and $[f]_*\mathcal{E}_Y = \mathcal{E}_X$ (i.e., if and only if $|X_{[f]}|_{([f])^{\times_{[f]}}_*\mathcal{E}_Y} = X$).

Proof. (\Rightarrow): Consider the push-out square

$$\begin{array}{ccc} Y & \xrightarrow{[f]} & X \\ [f] \downarrow & & \downarrow [e_1] \\ X & \xrightarrow{[e_2]} & X \amalg_Y X \end{array}$$

(in **Crs**). Fix a representative coarse map $f: Y \rightarrow X$. As a set, one may take $X \amalg_Y X := X_1 \sqcup X_2$ (disjoint union of sets) where $X_1 := X_2 := X$, with coarse structure

$$\langle \mathcal{E}_{X_1}, \mathcal{E}_{X_2}, \{(f_1 \times f_2)(F) : F \in \mathcal{E}_Y\} \rangle_{X_1 \sqcup X_2},$$

where $\mathcal{E}_{X_j} := \mathcal{E}_X \subseteq \wp((X_j)^{\times 2})$ and $f_j := f: Y \rightarrow X = X_j$, for $j = 1, 2$. As set maps, one may take e_1, e_2 to be the two inclusions.

If $X_{[f]} \neq X$, then there exists $x_0 \in X$ not connected to any $f(y)$, $y \in Y$. The entourage $\{1_{x_0}\} \in \mathcal{E}_X$ then shows that e_1 is not close to e_2 , hence $[e_1] \neq [e_2]$ while $[e_1] \circ [f] = [e_2] \circ [f]$ so $[f]$ is not epi. Similarly, if $E \in \mathcal{E}_X \setminus [f]_*\mathcal{E}_Y$, then one can show that $(e_1 \times e_2)(E)$ is not an entourage of $X \amalg_Y X$, hence again $[f]$ is not epi.

(\Leftarrow): It suffices to show that $|X_{[f]}|_{([f])^{\times_{[f]}}_*\mathcal{E}_Y} = X$ implies that $[e_1] = [e_2]$ in the push-out square considered above. If $|X_{[f]}|_{([f])^{\times_{[f]}}_*\mathcal{E}_Y} = X$, then every entourage of $[f]_*\mathcal{E}_Y$ is a subset of an entourage of the form $E_1 \circ f^{\times 2}(F) \circ E_2$ for $F \in \mathcal{E}_Y$ and $E_1, E_2 \in \mathcal{E}_X$. Thus if $[f]_*\mathcal{E}_Y = \mathcal{E}_X$, given $E \in \mathcal{E}_X$ choose F, E_1 , and E_2 so that $E \subseteq E_1 \circ f^{\times 2}(F) \circ E_2$, and then

$$(e_1 \times e_2)(E) \subseteq E_1 \circ (f_1 \times f_2)(F) \circ E_2$$

(where we now consider $E_j \in \mathcal{E}_{X_j} = \mathcal{E}_X$ for $j = 1, 2$) is an entourage of $X \amalg_Y X$. Thus e_1 is close to e_2 as required. \square

Corollary 3.10.7. For any $[f]: Y \rightarrow X$, the canonical arrow

$$[\tilde{f}]: Y \rightarrow |X_{[f]}|_{([f])^{\times_{[f]}}_*\mathcal{E}_Y}$$

is epi in **Crs**.

Corollary 3.10.8. Suppose $\mathcal{E}, \mathcal{E}'$ are coarse structures on a set X with $\mathcal{E}' \subseteq \mathcal{E}$. If every unital subspace of $|X|_{\mathcal{E}}$ is a unital subspace of $|X|_{\mathcal{E}'}$, then the class $[q]$ of the “identity” map

$$q: |X|'_{\mathcal{E}} \rightarrow |X|_{\mathcal{E}}$$

is epi in **Crs**.

Proof. Trivially, $(|X|_{\mathcal{E}})_{[q]} = |X|_{\mathcal{E}}$. We have that

$$[q]_*\mathcal{E}' = \langle \mathcal{E}' \rangle_{|X|_{\mathcal{E}}}$$

is an ideal of \mathcal{E} ; we must prove equality, so suppose $E \in \mathcal{E}$. Then $1_{E \cdot X}$ is in \mathcal{E} hence also in \mathcal{E}' , so

$$E = 1_{E \cdot X} \circ E$$

is in $[q]_* \mathcal{E}'$, as required. \square

Definition 3.10.9. Suppose $[f]: Y \rightarrow X$. Denote $\text{Coim}[f] := |X_{[f]}|_{([f]|^{X_{[f]}})_* \mathcal{E}_Y}$ and $\text{coim}[f] := [\tilde{f}]: Y \rightarrow \text{Coim}[f]$, where $[\tilde{f}]$ is defined as above.

Theorem 3.10.10. For any $[f]: Y \rightarrow X$, the quotient object of Y represented by the arrow $\text{coim}[f]: Y \twoheadrightarrow \text{Coim}[f]$ is the (categorical) coimage of $[f]$ in \mathbf{Crs} .

Proof. Suppose $[f]$ also factors as $Y \xrightarrow{[h]} Z \xrightarrow{[g]} X$ where $[h]$ is epi, so that $Z_{[h]} = Z$ and $\mathcal{E}_Z = [h]_* \mathcal{E}_Y$. We must show that there is a unique $[\bar{g}]: Z \rightarrow \text{Coim}[f]$ such that $\text{coim}[f] = [\bar{g}] \circ [h]$.

Pick representative coarse maps $g: Z \rightarrow X$ and $h: Y \rightarrow Z$. We may then take $f := g \circ h$ as a representative for $[f]$. Since $Z_{[f]} = Z$ and $[g] \circ [h] = [f]$, it follows that g has set image contained in $X_{[f]}$. Thus we may put $\bar{g} := g|^{X_{[f]}}$ as a set map $Z \rightarrow X_{[f]} = \text{Coim}[f]$. \bar{g} is a coarse map: It is locally proper since $g = \alpha \circ \bar{g}$ is locally proper. Since $\mathcal{E}_Z = [h]_* \mathcal{E}_Y$, every entourage of Z is contained in one of the form $G_1 \circ h^{\times 2}(F) \circ G_2$, for $F \in \mathcal{E}_Y$, $G_1, G_2 \in \mathcal{E}_Z$. For such an entourage,

$$\bar{g}^{\times 2}(G_1 \circ h^{\times 2}(F) \circ G_2) \subseteq g^{\times 2}(G_1) \circ (g \circ h)^{\times 2}(F) \circ g^{\times 2}(G_2)$$

is in $([f]|^{X_{[f]}})_* \mathcal{E}_Y$ since $g^{\times 2}(G_1), g^{\times 2}(G_2) \in \mathcal{E}_X$ (and g has set image in $X_{[f]}$) and $(g \circ h)^{\times 2}(F) = f^{\times 2}(F)$. Thus \bar{g} is coarse. From the uniqueness assertion of Proposition 3.10.5 (or, since $\tilde{f} = \bar{g} \circ h$), we get that $\text{coim}[f] = [\bar{g}] \circ [h]$. Uniqueness of $[\bar{g}]$ follows immediately from the hypothesis that $[h]$ is epi. \square

3.11. Monic and epi arrows. I do not know if \mathbf{Crs} is a *balanced* category, i.e., whether every arrow in \mathbf{Crs} which is both monic and epi is an isomorphism (the converse is always true, of course). To show that a monic and epi $[f]: Y \rightarrow X$ is an isomorphism one must show that there is an inverse $[f]^{-1}: X \rightarrow Y$. When X is unital, this is fairly straightforward (see below), but I do not know how to prove it when X is not.

Theorem 3.11.1. If $[f]: Y \rightarrow X$ is monic and epi in \mathbf{Crs} and X is a unital coarse space, then $[f]$ is an isomorphism in \mathbf{Crs} .

Proof. Fix a representative coarse map $f: Y \rightarrow X$. Since $[f]$ is epi, by Proposition 3.10.6, $X_{[f]} = X$ and

$$[f]_* \mathcal{E}_Y := \langle\langle f_* \mathcal{E}_Y \rangle\rangle_X = \mathcal{E}_X.$$

Then every entourage $E_0 \in \mathcal{E}_X$ is contained in one of the form $E_1 \circ E_2 \circ E_3$, where $E_1, E_3 \in \mathcal{E}_X$ and $E_2 \in f_* \mathcal{E}_Y$. Every $E_2 \in f_* \mathcal{E}_Y$ is contained in an entourage of the form

$$(f^{\times 2}(F_2^1) \circ \dots \circ f^{\times 2}(F_2^N)) \cup \bigcup_{j \in I} (K_j \times K'_j),$$

where $F_2^1, \dots, F_2^N \in \mathcal{E}_Y$ (some $N \geq 0$), J is the set of connected components of X , and K_j, K'_j are finite subsets of j for each $j \in J$. Since $X_{[f]} = X$ (and $f^{\times 2}(F_2^k) \in \mathcal{E}_X$ for $k = 2, \dots, N$), it follows that every $E \in \mathcal{E}_X$ is contained in a some entourage

$$E_0 \circ f^{\times 2}(F_0) \circ E'_0,$$

where $E_0, E'_0 \in \mathcal{E}_X$ and $F_0 \in \mathcal{E}_Y$.

We specialize the above discussion to the case $E = 1_X$ which is in \mathcal{E}_X by unitality. Fix $E_0, E'_0 \in \mathcal{E}_X$ and $F_0 \in \mathcal{E}_Y$, so that $1_X \subseteq E_0 \circ f^{\times 2}(F_0) \circ E'_0$. Define a set map $e: X \rightarrow Y$ as follows. For each $x \in X$, there are $x', x'' \in X$ and $y', y'' \in Y$ such that $(x, x') \in E_0$, $(x'', x) \in E'_0$, $f(y') = x'$, $f(y'') = x''$, and $(y', y'') \in F_0$; choosing such a $y'' \in Y$ in particular, put $e(x) := y''$.

We must verify that (any) $e: X \rightarrow Y$ as constructed above is a coarse map. Local properness: X is unital, so e is locally proper if and only if it is proper. For any $y \in Y$, $e^{-1}(\{y\}) \subseteq (E_0 \circ f^{\times 2}(F_0)) \cdot \{f(y)\}$ is finite, since $E_0 \circ f^{\times 2}(F_0) \in \mathcal{E}_X \subseteq \mathcal{E}_{|X|_1}$ satisfies the properness axiom. e preserves entourages: Fix $E \in \mathcal{E}_X$ and put $F := e^{\times 2}(E)$. Since $[f]$ is monic, by Proposition 3.9.5,

$$\mathcal{E}_Y = [f]^* \mathcal{E}_X := \mathcal{E}_{\text{Term}(Y)} \cap f^* \mathcal{E}_X.$$

Since e is (locally) proper, F satisfies the properness axiom; since the image of e is contained in the unital subspace $Y \cdot F_0$ of Y , it then follows that $F \in \mathcal{E}_{\text{Term}(Y)}$ and hence also that f is locally proper for F . To show that $F \in f^* \mathcal{E}_X$, it only remains to show that $f^{\times 2}(F) \in \mathcal{E}_X$: Since

$$G_0 := (\text{id}_X \times (f \circ e))(1_X) \subseteq E_0 \circ f^{\times 2}(F_0)$$

is in \mathcal{E}_X ,

$$f^{\times 2}(F) = (f \circ e)^{\times 2}(E) \subseteq (G_0)^T \circ E \circ G_0$$

is also in \mathcal{E}_X .

Since $G_0 \in \mathcal{E}_X$, we also get that $f \circ e$ is close to id_X , i.e., $[f \circ e] = [f] \circ [e]$ is the identity arrow $[\text{id}_X]$ of X in **Crs**. Since $[e]$ is monic (and $[f] \circ [e] \circ [f] = [f] = [f] \circ [\text{id}_Y]$), it also follows that $[e] \circ [f] = [\text{id}_Y]$. Thus $[e] = [f]^{-1}$, as required. \square

Corollary 3.11.2. *If $[f]: Y \rightarrow X$ is monic and epi in **Crs** and X is coarsely equivalent to a unital coarse space, then $[f]$ is an isomorphism in **Crs**.*

The problem with the above Corollary, of course, is that I do not know when a coarse space is coarsely equivalent to a unital one. If $\iota: X' \hookrightarrow X$ is the inclusion of a subspace of X into X , then $[\iota]$ is monic (and $\text{Im}[\iota] = X'$), so $\text{coim}[\iota]: X' \rightarrow \text{Coim}[\iota]$ is both monic and epi. (If X is connected and X' nonempty, $\text{Coim}[\iota]$ is just the set X equipped with the coarse structure of entourages in \mathcal{E}_X “supported near X' ”.) However, I do not know when $\text{coim}[\iota]$ is a coarse equivalence.

More generally, for any $[f]: Y \rightarrow X$, the natural arrow $Y \rightarrow \text{Im}[f]$ is epi (either use Proposition 3.10.6, or the fact that **Crs** has equalizers and, e.g., [18, Ch. I Prop. 10.1]) and hence there is a natural epi arrow $[\gamma]: \text{Im}[f] \rightarrow \text{Coim}[f]$ through which $\text{im}[f]: \text{Im}[f] \rightarrow X$ factors; as $\text{im}[f]$ is monic, $[\gamma]$ must also be monic. (One may dually show that the natural arrow $\text{Coim}[f] \rightarrow X$ is monic, but this yields the same arrow $[\mu]$.) Of course, I do not know when $[\mu]$ is an isomorphism. But when it is an isomorphism, one can, in a coarsely invariant way,

describe the image of $[f]$ as a subset of X with a certain coarse structure. This would be an appealing “generalization” of the following, which is not coarsely invariant in the desired sense.

Proposition 3.11.3. *If $f: Y \rightarrow X$ is a coarse map and Y is unital, then $\text{Im}[f] = f(Y)$ (where $f(Y)$ is the subspace of X determined by the set image of f) as subobjects of X in \mathbf{Crs} .*

Proof. If Y is unital, $X' := f(Y)$ is also unital. The range restriction $f|^{X'}: Y \rightarrow X'$ is a coarse map, and $[f]^* \mathcal{E}_X = [f|^{X'}]^* \mathcal{E}_{X'}$ hence $\text{Im}[f] = \text{Im}[f|^{X'}]$. Using this equality, we get $\text{im}[f] = [\iota] \circ \text{im}[f|^{X'}]$, where $\iota: X' \hookrightarrow X$ is the inclusion. But it is easy to check that $\mathcal{E}_{X'} := \mathcal{E}_X|_{X'} = [f|^{X'}]_* \mathcal{E}_Y$, so $[f|^{X'}]$ is epi. Hence $\text{im}[f|^{X'}]: \text{Im}[f] = \text{Im}[f|^{X'}] \rightarrow X'$ is both monic and epi, hence an isomorphism by Theorem 3.11.1. \square

3.12. Quotients of coarse spaces. We now discuss a notion of quotient coarse spaces in \mathbf{Crs} . The quotient spaces below are not the most general possible; rather, they appear to be a special case of a more general notion (of quotients by *coarse equivalence relations*). However, I have not fully explored the more general notion, and so I leave it to a future paper.

Suppose \mathcal{C} is a category with zero object 0 (e.g., an abelian category), i.e., 0 is both initial and terminal. Given an arrow $f: Y \rightarrow X$ (often taken to be monic) in \mathcal{C} , a standard way of defining the quotient, denoted $X/f(Y)$, is as the push-out $X \amalg_Y 0$ (assuming it exists); i.e., $X/f(Y)$ fits into a push-out square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X/f(Y) \end{array} .$$

The quotient $X/f(Y)$ comes equipped with an arrow $X \rightarrow X/f(Y)$ and, in the above case, also an arrow $0 \rightarrow X/f(Y)$.

In an abelian category, $X/f(Y)$ is by definition just the cokernel of f . If $\mathcal{C} = \mathbf{Set}_*$ or \mathbf{Top}_* (pointed sets or spaces), then one has $0 = *$ (a one-point set/space) and $X/f(Y)$ is (isomorphic to) just X with the image of f collapsed to the base point. The situation in \mathbf{Set} or \mathbf{Top} is slightly more complicated: If $Y \neq \emptyset$, one can again take the push-out $X/f(Y) := X \amalg_Y *$. However, if $Y = \emptyset$, then $X/f(Y) \cong X$; one should instead take $X/f(Y) := X \amalg_Y \emptyset$. In other words, one takes $X/f(Y) := X \amalg_Y \tilde{Y}$, where \tilde{Y} universally terminates Y (see Example 3.8.5). This is exactly what we do in the coarse categories.

Definition 3.12.1. Suppose $[f]: Y \rightarrow X$ (in \mathbf{Crs}). The *quotient coarse space* $X/[f](Y)$ is the push-out $X \amalg_Y \text{Term}(Y)$ in \mathbf{Crs} , i.e., $X/[f](Y)$ fits into a push-out

square

$$\begin{array}{ccc} Y & \xrightarrow{[f]} & X \\ [\tau_Y] \downarrow & & [q] \downarrow \\ \text{Term}(Y) & \xrightarrow{[f]/[f]} & X/[f](Y) \end{array} .$$

If $Y \subseteq X$ is a subspace, we will write $X/[Y] := X/[\iota(Y)]$, where $\iota: Y \hookrightarrow X$ is the inclusion.

The justification for our notation is the following.

Proposition 3.12.2. *For any $[f]: Y \rightarrow X$, the quotient coarse space $X/[f](Y)$ and the natural map $X \rightarrow X/[f](Y)$ only depend on the image of $[f]$.*

Proof. $[f]$ factorizes canonically as

$$Y \xrightarrow{[\beta]} [f](Y) \xrightarrow{\text{im}[f]} X$$

(Proposition 3.9.4 and Definition 3.9.7). Thus $X/[f](Y)$ is also the colimit of the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{[\beta]} & [f](Y) & \xrightarrow{\text{im}[f]} & X \\ [\tau_Y] \downarrow & & [\tau_{[f](Y)}] \downarrow & & \\ \text{Term}(Y) & \xrightarrow{[\text{Term}(\beta)]} & \text{Term}([f](Y)), & & \end{array}$$

and hence also of the cofinal subdiagram obtained by deleting Y and $\text{Term}(Y)$. \square

The coarse categories have all push-outs and we have seen how to describe them concretely; the standard construction would take $X/[f](Y)$ to be, as a set, the disjoint union of X , Y , and $\text{Term}(Y)$. Taking a representative coarse map $f: Y \rightarrow X$, we have two “smaller” descriptions of the quotient:

- (i) Take $X/[f](Y) := X \sqcup \text{Term}(Y)$ as a set with the coarse structure generated by \mathcal{E}_X , $\mathcal{E}_{\text{Term}(Y)}$, and $\{(f \times \tau_Y)(F): F \in \mathcal{E}_Y\}$, where we consider $\text{Term}(Y)$ and X as subsets of $X \sqcup \text{Term}(Y)$. (This is a particular instance of a “smaller” construction of push-outs in **Crs**.)
- (ii) Take $X/[f](Y) := X$ as a set with the coarse structure generated by \mathcal{E}_X and $f_*\mathcal{E}_{\text{Term}(Y)}$, where we treat f as a set map $\text{Term}(Y) = Y \rightarrow X$.

Using the second description above and applying Corollary 3.10.8 (the left and right supports of entourages in $f_*\mathcal{E}_{\text{Term}(Y)}$ are already unital subspaces of X), we immediately get the following.

Proposition 3.12.3. *For any $[f]: Y \rightarrow X$, $X/[f](Y)$ is a quotient of X in the categorical sense (i.e., the natural map $[q]: X \rightarrow X/[f](Y)$ is epi).*

3.13. Restricted coarse categories. The lack of restriction on the size of coarse spaces (other than that imposed by the choice of universe) may be somewhat bothersome, and moreover prevent **Crs** from having a terminal object. It is tempting to restrict the cardinality of coarse spaces, i.e., consider the full subcategory of **Crs** of the coarse spaces of cardinality at most κ , for some fixed, small (probably infinite) cardinal κ . This is not the correct thing to do: First, one would no longer have all small limits and colimits (though as long as κ is infinite one have all finite limits and colimits). Second, and more importantly, it would bar constructions involving the set of (set) functions $Y \rightarrow X$ ($\#X, \#Y \leq \kappa$) which will be important in [13].

A better way to proceed is to consider the full subcategory of **Crs** of coarse spaces X for which there exists a coarse map $X \rightarrow R$, where $R := \text{Term}(R_0)$ for some fixed R_0 . (Of particular interest is the case when R_0 is a unital coarse space of some infinite cardinality κ , in which case $R = |R_0|_1$ only depends on κ up to coarse equivalence.)

We will first discuss this in full generality, using terminology from the beginning of §3.8. In the following, suppose C is some category and that is some object which \tilde{X} terminates any object (e.g., itself) in C .

Definition 3.13.1. The \tilde{X} -restriction $C_{\preceq \tilde{X}}$ of C is the full subcategory of C consisting of all the objects Y in C such that there exists some (unique) arrow $Y \rightarrow \tilde{X}$.

In other words, $C_{\preceq \tilde{X}}$ consists of all objects which are terminated by \tilde{X} . Equivalently, one may consider the comma category $(C \downarrow \tilde{X})$. It is easy to check that the range restricted projection functor $(C \downarrow \tilde{X}) \rightarrow C_{\preceq \tilde{X}}$ is an isomorphism of categories.

Let $I: C_{\preceq \tilde{X}} \rightarrow C$ denote the inclusion functor. When a *nonzero* limit $C_{\preceq \tilde{X}}$ already exists in C , the limits are the same. More precisely, we have the following.

Proposition 3.13.2. Suppose $\mathcal{F}: \mathcal{J} \rightarrow C_{\preceq \tilde{X}}$, where \mathcal{J} is nonempty. If the limit $C\text{-Lim}(I \circ \mathcal{F})$ exists, then

$$C_{\preceq \tilde{X}}\text{-Lim } \mathcal{F} = C\text{-Lim}(I \circ \mathcal{F});$$

i.e., the limit of \mathcal{F} in $C_{\preceq \tilde{X}}$ exists and any limiting cone in C gives a limiting cone in $C_{\preceq \tilde{X}}$.

Proof. The nonemptiness of \mathcal{J} ensures that the object $C\text{-Lim}(I \circ \mathcal{F})$ is in $C_{\preceq \tilde{X}}$ (since it must map to some object of $C_{\preceq \tilde{X}}$, hence to \tilde{X}). The rest follows easily, since the inclusion functor I is fully faithful. \square

The following is trivial.

Proposition 3.13.3. \tilde{X} is a terminal object (i.e., zero limit) in $C_{\preceq \tilde{X}}$.

Thus $C_{\preceq \tilde{X}}$ has all the limits that C does (to the extent that this makes sense), but also has a terminal object, which C may not have. However, C may have a terminal object which is not isomorphic to \tilde{X} (in which case $C_{\preceq \tilde{X}}$ is a proper subcategory of C), so the inclusion functor I may not preserve limits.

The result dual to Proposition 3.13.4 is true without the nonemptiness criterion.

Proposition 3.13.4. *Suppose $\mathcal{F} : \mathcal{J} \rightarrow C_{\leq \tilde{X}}$. If the colimit $\mathcal{C}\text{-Colim}(I \circ \mathcal{F})$ exists, then*

$$C_{\leq \tilde{X}}\text{-Colim } \mathcal{F} = \mathcal{C}\text{-Colim}(I \circ \mathcal{F}).$$

Proof. If $\mathcal{C}\text{-Colim}(I \circ \mathcal{F})$ exists, then it maps to \tilde{X} since there is a (unique) cone $\mathcal{J} \rightarrow \tilde{X}$; thus the colimiting cone is actually in $C_{\leq \tilde{X}}$ and is universal since I is fully faithful. \square

Now, we return to our coarse context. Suppose $R := \text{Term}(R_0)$ for some coarse space R_0 . The R -restricted coarse category $\mathbf{Crs}_{\leq R}$ is, as the notation indicates, the R -restriction of \mathbf{Crs} . We similarly get R -restricted connected and connected, nonempty coarse categories. We refer to the above categories collectively (i.e., for all R and the various cases) as the *restricted coarse categories*.

Theorem 3.13.5. *The restricted coarse categories have all (small) limits and colimits.*

Proof. This follows immediately from Theorems 3.5.11 and 3.7.6, and Propositions 3.13.2 and 3.13.4. \square

One can also check that all the earlier facts on monics and images, epis and coimages, quotients, etc. hold in the restricted coarse categories.

4. TOPOLOGY AND COARSE SPACES

Our coarse spaces are discrete, as opposed to the more standard definition of *proper coarse spaces* which allows coarse spaces to carry topologies and thus has different properness requirements (see the works of Roe, e.g., [29, Def. 2.22]). Our aim here is not to provide a general discussion of topological coarse spaces but to provide a means from going from Roe's *proper coarse spaces* to our (discrete) coarse spaces.

We will use the terms *compact* and *locally compact* in the sense of Bourbaki [17, Ch. I §9], including the Hausdorff condition; in fact, all spaces will be Hausdorff unless otherwise stated. Throughout, X and Y will denote paracompact, locally compact topological spaces. Recall that a subset K of a space X is *relatively compact* if it is contained in some compact subspace of X . (If X is Hausdorff, K is relatively compact if and only if \bar{K} is compact.)

4.1. Roe coarse spaces. We will diverge from the standard terminology to avoid confusion with our previously defined terms. *Roe coarse spaces* will be what are usually called proper coarse spaces. Let us recall these definitions (compare Definitions 1.3.1 and 1.3.6).

Definition 4.1.1 (see, e.g., [29, Def. 2.1]). A subset $E \subseteq X^{\times 2}$ satisfies the *Roe properness axiom* if $E \cdot K$ and $K \cdot E$ are relatively compact subsets of X for all (relatively) compact $K \subseteq X$.

Definition 4.1.2 (see, e.g., [29, Def. 2.22]). A *Roe coarse structure* on X is a subset $\mathcal{R}_X \subseteq \wp(X^{\times 2})$ such that:

- (i) each $E \in \mathcal{R}_X$ satisfies the Roe properness axiom;
- (ii) \mathcal{R}_X is closed under the operations of addition, multiplication, transpose, and the taking of subsets;
- (iii) if $K \subseteq X$ is *bounded* in the sense that $K^{\times 2} \in \mathcal{R}_X$, then K is relatively compact; and
- (iv) there is a neighbourhood (with respect to the product topology on $X^{\times 2}$ of the unit (i.e., diagonal) 1_X which is in \mathcal{R}_X .

A *Roe coarse space* is a paracompact, locally compact space X equipped with a Roe coarse structure \mathcal{R}_X on X .

(iv) implies Roe coarse spaces are always unital (in the obvious sense; see Definition 1.4.1) and that any Roe coarse space X has an open cover $\mathcal{U} \subseteq \wp(X)$ which is *uniformly bounded* in the sense that $\bigcup_{U \in \mathcal{U}} U^{\times 2}$ is in \mathcal{R}_X . Paracompactness implies that this cover can be taken to be locally finite. The local compactness requirement is redundant, since it is implied by (iii) and (iv).

Definition 4.1.3. A continuous map $f: Y \rightarrow X$ between locally compact spaces is *topologically proper* if $f^{-1}(K)$ is compact for every compact $K \subseteq X$. More generally, also say that a (not necessarily continuous) map $f: Y \rightarrow X$ between locally compact spaces is *topologically proper* if $f^{-1}(K)$ is relatively compact for every relatively compact $K \subseteq X$.

Definition 4.1.4 (see, e.g., [29, Def. 2.21 and 2.14]). A (not necessarily continuous) map $f: Y \rightarrow X$ between Roe coarse spaces is a *Roe coarse map* if it is topologically proper and *preserves entourages* in the sense that $f^{\times 2}(F) \in \mathcal{R}_X$ for all $F \in \mathcal{R}_Y$. (Roe coarse maps are usually called *proper coarse maps*.) Roe coarse maps $f, f': Y \rightarrow X$ are *close* if $(f \times f')(1_Y) \in \mathcal{R}_X$ (or equivalently if $(f \times f')(F) \in \mathcal{R}_X$ for all $F \in \mathcal{R}_Y$).

We get an obvious *Roe precoarse category* **RoePCrs** with objects all (small) Roe coarse spaces and arrows Roe coarse maps, and a quotient *Roe coarse category* **RoeCrS** with the same objects but whose arrows are closeness classes of Roe coarse maps. *Roe coarse equivalences* are Roe coarse maps which represent isomorphisms in **RoeCrS**.

4.2. Discretization of Roe coarse spaces. We now provide a way of passing from Roe coarse spaces to our (discrete) coarse spaces.

Definition 4.2.1. A set $E \in \wp(X^{\times 2})$ satisfies the *topological properness axiom* (with respect to the topology of X) if, for all compact subspaces $K \subseteq X$, $(\pi_1|_E)^{-1}(K)$ and $(\pi_2|_E)^{-1}(K)$ are finite.

Since all our spaces are Hausdorff hence T_1 , the topological properness axiom implies the (discrete) properness axiom (Definition 1.3.1).

The following is easy to check.

Proposition 4.2.2. A set $E \in \wp(X^{\times 2})$ satisfies the topological properness axiom if and only if E is a (closed) discrete subset of $X^{\times 2}$ and the restricted projections $\pi_1|_E, \pi_2|_E: E \rightarrow X$ are topologically proper maps.

Remark 4.2.3. We provide only a means from passing from Roe coarse spaces to our coarse spaces and not a complete discussion of “topological coarse spaces” since the topological properness axiom does not encompass the axioms of Definition 4.1.2 (iv) in particular). We would like not just a direct translation of Roe’s definition to our setting, but a proper generalization: First, we would like to allow nonunital topological coarse spaces. Second, we do not want to impose local compactness for two (possibly related) reasons: (i) The “topological coarse category” should have all colimits (including infinite ones). In particular, we are interested in “large” simplicial complexes which may not be locally finite. (ii) We wish to be able to analyze Hilbert space and other Banach spaces directly as coarse spaces. This seems especially relevant as methods involving uniform (i.e., coarse) embeddings into such spaces have gained prominence in recent years (e.g., in [32], Yu shows that the Coarse Baum–Connes Conjecture is true for metric spaces of bounded geometry which uniformly embed in Hilbert space).

Instead of requiring that spaces be paracompact and locally compact, we should probably require that spaces be *compactly generated* (i.e., be weak Hausdorff k -spaces). The topological properness axiom makes sense for such spaces (weak Hausdorffness still implies the T_1 condition), but the problem of translating axioms (iii) and (iv) becomes more complicated. Moreover, in the compactly generated case, there are different, inequivalent definitions for “topological properness” (whereas they all agree in the locally compact case; see, e.g., [17, Ch. I §10]), though perhaps one could still use Definition 4.1.3 verbatim. In that case, the above Proposition remains true so long as $X^{\times 2}$ is given the categorically appropriate topology, namely the k -ification of the standard product topology. We leave these problems to a future paper [14].

Compare the following, which is easy, with Proposition 1.3.5.

Proposition 4.2.4. *If $E, E' \in \wp(X^{\times 2})$ satisfy the topological properness axiom, then $E + E'$, $E \circ E'$, E^T , and all subsets of E satisfy the topological properness axiom. Also, all singletons $\{e\}$, $e \in X^{\times 2}$, and hence all finite subsets of $X^{\times 2}$ satisfy the properness axiom. Consequently,*

$$\mathcal{E}_{|X|_\tau} := \{E \in \mathcal{E}_{|X|_\tau} \subseteq \wp(X^{\times 2}) : E \text{ satisfies the topological properness axiom}\}$$

is a coarse structure on the set X (in the sense of Definition 1.3.6).

Definition 4.2.5. The *discretization* of a Roe coarse space X is the coarse space $\text{Disc}(X) := X$ as a set with the coarse structure

$$\mathcal{E}_{\text{Disc}(X)} := \mathcal{R}_X \cap \mathcal{E}_{|X|_\tau}$$

(consisting of all elements of \mathcal{R}_X which satisfy the topological properness axiom).

It is easy to check that $\mathcal{E}_{\text{Disc}(X)}$ is in fact a coarse structure on the set X . Unless X is discrete, the coarse space $\text{Disc}(X)$ is not unital, even though the Roe coarse space X is.

Proposition 4.2.6. *If $f: Y \rightarrow X$ is a Roe coarse map, then the set map $\text{Disc}(f) := f$ is coarse as a map $\text{Disc}(Y) \rightarrow \text{Disc}(X)$.*

Proof. The only thing to check is that if f (not necessarily continuous) is topologically proper and $F \subseteq Y^{\times 2}$ satisfies the topological properness axiom, then $E := f^{\times 2}(F) \subseteq X^{\times 2}$ also satisfies the topological properness axiom. This follows since

$$E \cdot K \subseteq f(F \cdot f^{-1}(K))$$

and f is topologically proper (and similarly symmetrically). \square

Since, trivially, $\text{Disc}(f \circ g) = \text{Disc}(f) \circ \text{Disc}(g)$, we get the following.

Corollary 4.2.7. *Disc is a functor from the Roe precoarse category $\mathbf{RoePCrs}$ to the precoarse category \mathbf{PCrs} .*

Disc is coarsely invariant in the following way, which yields a canonical functor $[\text{Disc}]: \mathbf{RoeCrs} \rightarrow \mathbf{Crs}$ between the closeness quotients. (We continue to write $\text{Disc}(X)$ instead of $[\text{Disc}](X)$ for Roe coarse spaces.)

Proposition 4.2.8. *If Roe coarse maps $f, f': Y \rightarrow X$ are close, then*

$$\text{Disc}(f), \text{Disc}(f'): \text{Disc}(Y) \rightarrow \text{Disc}(X)$$

are close coarse maps.

Proof. The result follows easily from the following fact (which is also easy): If $f, f': Y \rightarrow X$ are topologically proper and $F \subseteq Y^{\times 2}$ satisfies the topological properness axiom, then $(f \times f')(F) \subseteq X^{\times 2}$ also satisfies the topological properness axiom. \square

Corollary 4.2.9. *If $f: Y \rightarrow X$ is a Roe coarse equivalence, then $\text{Disc}(f): \text{Disc}(Y) \rightarrow \text{Disc}(X)$ is a coarse equivalence.*

4.3. Properties of the discretization functors. Let $\mathbf{DRoePCrs} \subseteq \mathbf{RoePCrs}$ and $\mathbf{DRoeCrs} \subseteq \mathbf{RoeCrs}$ be the full subcategories of *discrete* Roe coarse spaces (call them the *discrete Roe precoarse* and *coarse categories*, respectively). On the discrete subcategories, Disc and $[\text{Disc}]$ are fully faithful.

Proposition 4.3.1. *If X, Y are Roe coarse spaces with Y discrete, then the map*

$$\text{Disc}_{Y,X}: \text{Hom}_{\mathbf{RoePCrs}}(Y, X) \rightarrow \text{Hom}_{\mathbf{PCrs}}(\text{Disc}(Y), \text{Disc}(X))$$

is a bijection. Hence, in particular, the restriction of Disc to $\mathbf{DRoePCrs}$ (which actually maps into \mathbf{UPCrs}) is a fully faithful functor.

Proof. $\text{Disc}_{Y,X}$ is trivially injective, so it only remains to show surjectivity. Suppose $f: \text{Disc}(Y) \rightarrow \text{Disc}(X)$ is a coarse map. If $K \subseteq X$ is relatively compact, then

$$f^{-1}(K) = f^{-1}(f^{\times 2}(1_Y) \cdot K)$$

is finite: since Y is discrete, $\text{Disc}(Y)$ is unital so $f^{\times 2}(1_Y) \in \mathcal{E}_{\text{Disc}(X)}$ satisfies the topological properness axiom (so $f^{\times 2}(1_Y) \cdot K$ is finite) and f is (discretely) globally proper. Thus f is topologically proper. Since Y is discrete, $\mathcal{E}_{\text{Disc}(Y)} = \mathcal{R}_Y$, so

f preserves entourages of \mathcal{R}_Y (of course, $\mathcal{E}_{\text{Disc}(X)} \subseteq \mathcal{R}_X$). Thus f is Roe coarse as a map $Y \rightarrow X$. \square

The unrestricted functor $\text{Disc}: \mathbf{RoePCrs} \rightarrow \mathbf{PCrs}$ is *not* full.

Example 4.3.2. Let $X := \mathbb{R}_+$ equipped with the Euclidean metric Roe coarse structure (see §5.1), and $Y := \mathbb{R}_+ \cup \{\infty\}$ be the one-point compactification of \mathbb{R}_+ equipped with the unique Roe coarse structure $\mathcal{R}_Y := \wp(Y^{\times 2})$ (which is also the metric Roe coarse structure for any metric which metrizes Y topologically). Define $f: Y \rightarrow X$ by

$$f(t) := \begin{cases} t & \text{if } t \in \mathbb{R}_+, \text{ and} \\ 0 & \text{if } t = \infty. \end{cases}$$

Then f is actually coarse as a map $\text{Disc}(Y) \rightarrow \text{Disc}(X)$. However, clearly f does not preserve entourages of \mathcal{R}_Y , hence does *not* define a Roe coarse map $Y \rightarrow X$. As a map $\text{Disc}(Y) \rightarrow \text{Disc}(X)$, f is close to any constant map $\text{Disc}(Y) \rightarrow \text{Disc}(X)$ (sending all of Y to some fixed element of X); every such constant map *does* define a Roe coarse map $Y \rightarrow X$.

Proposition 4.3.3. *If X, Y are Roe coarse spaces with Y discrete, then the map*

$$[\text{Disc}]_{Y,X}: \text{Hom}_{\mathbf{RoeCrs}}(Y, X) \rightarrow \text{Hom}_{\mathbf{Crs}}(\text{Disc}(Y), \text{Disc}(X))$$

is a bijection. Hence the restriction of $[\text{Disc}]$ to $\mathbf{DRoeCrs}$ (which actually maps into \mathbf{UCrs}) is fully faithful.

Proof. By the previous Proposition, $[\text{Disc}]_{Y,X}$ is surjective, so it only remains to show injectivity. Suppose $f, f': Y \rightarrow X$ are Roe coarse maps. If $\text{Disc}(f)$ is close to $\text{Disc}(f')$, then since $\text{Disc}(Y)$ is unital,

$$(f \times f')(1_Y) = (\text{Disc}(f) \times \text{Disc}(f'))(1_Y) \in \mathcal{E}_{\text{Disc}(X)} \subseteq \mathcal{R}_X,$$

so f is close to f' , as required. \square

If $X' \subseteq X$ is a *closed* subspace of a Roe coarse space, then the obvious *Roe subspace coarse structure* $\mathcal{R}_{X'} := \mathcal{R}_X|_{X'} := \mathcal{R}_X \cap \wp((X')^{\times 2})$ is actually Roe coarse structure on X' (this is not the case if X' is not closed), which makes X' into a *Roe coarse subspace* of X . The inclusion of any Roe coarse subspace into the ambient space is a Roe coarse map. The following result is well known.

Proposition 4.3.4. *For any Roe coarse space X , there is a (closed) discrete Roe coarse subspace $X' \subseteq X$ such that the inclusion $\iota: X' \rightarrow X$ is a Roe coarse equivalence.*

Proof. Fix a locally finite, uniformly bounded open cover \mathcal{U} of X by nonempty sets. For each $U \in \mathcal{U}$, pick a point $x'_U \in U$ and put $X' := \{x'_U: U \in \mathcal{U}\}$. Since \mathcal{U} is locally finite, it is easy to check that X' is closed and discrete.

Invoking the Axiom of Choice, fix a map $\kappa: X \rightarrow X'$ such that, for all $x \in X$, $\kappa(x) \in U$ for some $U \in \mathcal{U}$ such that $x \in U$. We may also ensure that $\kappa(x') = x'$ for all $x' \in X'$. κ is topologically proper: For any $x' \in X'$,

$$\kappa^{-1}(\{x'\}) \subseteq \bigcup_{\substack{U \in \mathcal{U}: \\ x' \in U}} U$$

which is a finite union of relatively compact sets, hence $\kappa^{-1}(\{x'\})$ is relatively compact (this suffices to show topological properness since X' is discrete). κ preserves entourages of X : Put

$$E_{\mathcal{U}} := \bigcup_{U \in \mathcal{U}} U^{\times 2} \in \mathcal{R}_X;$$

for any $E \in \mathcal{R}_X$,

$$\kappa^{\times 2}(E) \subseteq E_{\mathcal{U}} \circ E \circ E_{\mathcal{U}} \in \mathcal{R}_X,$$

hence $\kappa^{\times 2}(E) \in \mathcal{R}_X|_{X'}$, as required. Thus κ is a Roe coarse map.

Trivially, $\kappa \circ \iota = \text{id}_{X'}$. Finally, $\iota \circ \kappa$ is close to id_X : Letting $E_{\mathcal{U}}$ be as above, we have

$$(\kappa \times \text{id}_X)(1_X) \subseteq E_{\mathcal{U}} \in \mathcal{R}_X,$$

as required. \square

Remark 4.3.5. Though we do not so insist, Roe coarse maps are sometimes required to be Borel (see, e.g., [9, Def. 2.2]). In that case, the map κ used in the above proof may not suffice. However, if one insists that all Roe coarse spaces be, e.g., second countable, then one can construct κ to be Borel. Thus, as long as one so constrains the allowable Roe coarse spaces, the above Proposition remains true.

Corollary 4.3.6. *The inclusion functor $\mathbf{DRoeCrs} \hookrightarrow \mathbf{RoeCrs}$ is fully faithful and in fact an equivalence of categories.*

Theorem 4.3.7. *The functor $[\text{Disc}]: \mathbf{RoeCrs} \rightarrow \mathbf{Crs}$ is fully faithful.*

Proof. This is immediate upon combining Propositions 4.3.3 and 4.3.4. \square

Every unital coarse space (in our sense) becomes a Roe coarse space when it is given the discrete topology, with coarse maps between unital coarse spaces becoming Roe coarse maps. Thus \mathbf{UPCrs} and $\mathbf{DRoePCrs}$ are isomorphic as categories, and hence so too are \mathbf{UCrs} and $\mathbf{DRoeCrs}$.

Corollary 4.3.8. *Our unital coarse category \mathbf{UCrs} is equivalent to the Roe coarse category \mathbf{RoeCrs} , with the functor which sends a unital coarse space to the “identical” discrete Roe coarse space an equivalence of categories.*

5. EXAMPLES AND APPLICATIONS

As stated in the Introduction, we will not discuss even the standard applications of coarse geometry. We will first discuss a couple of basic examples which we will need later, namely proper metric spaces and continuous control, and then briefly examine a few things which arise from the categorical point of view (some of which are not obviously possible in standard, unital coarse geometry).

5.1. Proper metric spaces. Suppose that $(X, d) := (X, d_X)$ is a proper metric space (i.e., its closed balls are compact). We wish to produce a coarse space from X ; we have already discussed the discrete case in Example 1.3.14, and what follows is a generalization of that.

There is a well known way to produce a Roe coarse space $|X|_d^R$ from (X, d) (noting that properness implies local compactness, and metrizable implies paracompactness), taking the Roe coarse structure to be consist of the $E \subseteq X \times X$ satisfying inequality (1.3.15) of Example 1.3.14 (see, e.g., [29, Ex. 2.5]). One can then apply the discretization functor Disc to this Roe coarse space to obtain the (d) -metric coarse space $|X| := |X|_d$. More directly and entirely equivalently, $|X|_d$ has as entourages the $E \in \mathcal{E}_{|X|_\tau}$ (i.e., the E satisfying the topological properness axiom) which also satisfy the same inequality (1.3.15). As in the discrete case, we may also allow $d(x, x') = \infty$ (for $x \neq x'$), and $|X|_d$ is connected if and only if $d(x, x') < \infty$ for all x and x' . If $X' \subseteq X$ is a closed (topological) subspace, then the restriction of d to X' makes X' into a proper metric space; the subspace coarse structure on X' is the same as the coarse structure coming from the restricted metric.

Suppose (Y, d_Y) is another proper metric space. A (not necessarily continuous) map $f: Y \rightarrow X$ is Roe coarse as a map $|Y|_{d_Y}^R \rightarrow |X|_{d_X}^R$ if and only if it is topologically proper and

$$(5.1.1) \quad \sup\{d_X(f(y), f(y')) : y, y' \in Y \text{ and } d_Y(y, y') \leq r\} < \infty$$

for every $r \geq 0$. Since X, Y are proper metric spaces, f is topologically proper if and only if it is *metrically proper* in the sense that inverse images of metrically bounded subsets of X are metrically bounded in Y . Roe coarse maps $f, f': |Y|_{d_Y}^R \rightarrow |X|_{d_X}^R$ are close if and only if

$$(5.1.2) \quad \sup\{d_X(f(y), f'(y)) : y \in Y\} < \infty.$$

We must warn that there may be a map $f: Y \rightarrow X$ which is coarse (in our sense) as a map $|Y|_{d_Y} \rightarrow |X|_{d_X}$, yet does not satisfy (5.1.1). Similarly, there may be coarse maps $f, f': |Y|_{d_Y} \rightarrow |X|_{d_X}$ which are close but do not satisfy (5.1.2). Example 4.3.2, which shows that Disc is not full, exhibits both phenomena. In the former case, Theorem 4.3.7 shows that every coarse map $f': |Y|_{d_Y} \rightarrow |X|_{d_X}$ is close to some coarse map $f: |Y|_{d_Y} \rightarrow |X|_{d_X}$ which satisfies 5.1.1. (The corresponding statement in the latter case is trivial.) Alternatively, one may avoid both “problems” by considering only discrete, proper metric spaces (Proposition 4.3.1); every proper metric space is Roe coarsely equivalent to a discrete one (Proposition 4.3.4).

Remark 5.1.3 (see, e.g., [29, §1.3]). If X and Y are proper *length spaces*, then one can characterize the Roe coarse maps, and indeed Roe coarse equivalences, $Y \rightarrow X$ a bit more strictly: A map $f: Y \rightarrow X$ (not necessarily continuous) is Roe coarse if and only if it is (metrically/topologically) proper and *large-scale Lipschitz* in the sense that there exist constants $C > 0$ and $R \geq 0$ such that

$$d_X(f(y), f(y')) \leq C d_Y(y, y') + R$$

for all $y, y' \in Y$. f is a Roe coarse equivalence if and only if it is a *quasi-isometry* in that there are constants $c, C > 0$ and $r, R \geq 0$ such that

$$cd_Y(y, y') - r \leq d_X(f(y), f(y')) \leq Cd_Y(y, y') + R$$

for all $y, y' \in Y$ (evidently, one can always take $c = 1/C$ and $r = R$, as is conventional) and there is a constant $D \geq 0$ such that every point of X is within distance D of a point in the image of f .

One can replace the length space hypothesis with a weaker condition, but some hypothesis is necessary; for general metric spaces there are Roe coarse maps, and indeed Roe coarse equivalences, which are not large-scale Lipschitz. However, every proper large-scale Lipschitz map is evidently also Roe coarse, and every quasi-isometry is a coarse equivalence.

5.2. Continuous control. Most of the following originates from [1, 9], but see also, e.g., [29, §2.2]. In the following, all topological spaces will be assumed to be second countable and locally compact (and Hausdorff), whence paracompact. X and Y will always denote such spaces.

Definition 5.2.1. A *compactified space* is a (second countable, locally compact) topological space X equipped with a (second countable) compactification \bar{X} ; its *boundary* is the space $\partial X := \bar{X} \setminus X$.

The *continuously controlled Roe coarse structure* $\mathcal{R}_{|X|_{\partial X}^R}$ on X (for the compactification \bar{X} , or for the boundary ∂X) consists of the $E \subseteq X^{\times 2}$ such that

$$(5.2.2) \quad \bar{E} \subseteq X^{\times 2} \cup 1_{\partial X} \subseteq \bar{X}^{\times 2},$$

where $1_{\partial X}$ is the diagonal subset of $(\partial X)^{\times 2}$ and the closure is taken in \bar{X} (for the proof that this is a Roe coarse structure, see, e.g., [29, Thm. 2.27]). The associated coarse space (resulting from applying the discretization functor Disc to the above Roe coarse space $|X|_{\partial X}^R$) is the *continuously controlled coarse space* $|X|_{\partial X}$ (for the compactification \bar{X} , or for the boundary ∂X) whose entourages are the $E \in \mathcal{E}_{|X|_{\partial X}}$ (i.e., E satisfying the topological properness axiom) which also satisfy (5.2.2).

Remark 5.2.3. If X is compact (so $\bar{X} = X$ and $\partial X = \emptyset$), then $|X|_{\partial X} = |X|_0^C$ (i.e., X equipped with the initial connected coarse structure).

The following is standard.

Proposition 5.2.4. Suppose X, Y are compactified spaces. Any Roe coarse map $f: |Y|_{\partial Y}^R \rightarrow |X|_{\partial X}^R$ determines a canonical continuous map $\partial Y \rightarrow \partial X$ which we denote by $\partial[f]$. Moreover, Roe coarse maps $f, f': |Y|_{\partial Y}^R \rightarrow |X|_{\partial X}^R$ are close if and only if $\partial[f] = \partial[f']$ (which justifies our notation).

The “converse” is also true: Any set map $Y \rightarrow X$ (not necessarily continuous, but necessarily topologically proper) which “extends continuously” to a continuous map $\partial Y \rightarrow \partial X$ is Roe coarse as a map $|Y|_{\partial Y}^R \rightarrow |X|_{\partial X}^R$. This is essentially tautological, since the definition of “extends continuously” is exactly the definition of “is continuously controlled”.

Proof. Fix a Roe coarse map f . Given $y_\infty \in \partial Y$, define $(\partial[f])(y_\infty)$ as follows: By second countability, there is a sequence $(y_n)_{n=1}^\infty$ in Y which converges to y_∞ . Then the diagonal set $1_{\{y_n: n \in \mathbb{N}\}}$ is in $\mathcal{R}_{|Y|_{\partial Y}}^{\mathbb{R}}$, so $1_{\{f(y_n): n \in \mathbb{N}\}}$ must be in $\mathcal{R}_{|X|_{\partial X}}^{\mathbb{R}}$. By topological properness, the limit points of $(f(y_n))_{n=1}^\infty$ in \bar{X} (which exist by compactness) are all in $\partial X \subseteq \bar{X}$; in fact there is only one limit point which we call $(\partial[f])(y_\infty)$. Well-definedness follows from the observation that if $(y'_n)_{n=1}^\infty \subseteq Y$ (possibly a subsequence of $(y_n)_{n=1}^\infty$) also converges to y_∞ , then $1_{\{(y_n, y'_n): n \in \mathbb{N}\}} \in \mathcal{R}_{|Y|_{\partial Y}}^{\mathbb{R}}$ hence $1_{\{(f(y_n), f(y'_n)): n \in \mathbb{N}\}} \in \mathcal{R}_{|X|_{\partial X}}^{\mathbb{R}}$, so $(f(y'_n))_{n=1}^\infty$ and $(f(y_n))_{n=1}^\infty$ have the same limit points. To see that $\partial[f]$ is continuous, one proves sequential continuity (which suffices) using the obvious diagonal argument.

f (and similarly f') “extend continuously” to maps $\bar{Y} \rightarrow \bar{X}$: e.g.,

$$\bar{f}(y) := \begin{cases} f(y) & \text{if } y \in Y, \text{ and} \\ (\partial[f])(y) & \text{if } y \in \partial Y. \end{cases}$$

The second assertion then follows using the observation that, for any $F \in \mathcal{R}_{|Y|_{\partial Y}}^{\mathbb{R}}$,

$$\overline{(f \times f')(F)} = (\bar{f} \times \bar{f}')(\bar{F})$$

(closures $\bar{X}^{\times 2}$ and $\bar{Y}^{\times 2}$). □

Temporarily let \mathcal{C} be the category of second countable, compact spaces (and continuous maps). If $M \in \text{Obj}(\mathcal{C})$, $\mathbb{R}_+ \times M$ compactified with boundary M (so $\mathbb{R}_+ \times M$ is homeomorphic to $[0, 1] \times M$) is a compactified space. Then $M \mapsto |\mathbb{R}_+ \times M|_M^{\mathbb{R}}$ (on objects; $g \mapsto \text{id}_{\mathbb{R}_+} \times g$ on functions) defines a (Roe) coarsely invariant functor $O_{\text{top}}^{\mathbb{R}}: \mathcal{C} \rightarrow \mathbf{RoePCrs}$. By the above Proposition,

$$[O_{\text{top}}^{\mathbb{R}}] := \text{Quot} \circ O_{\text{top}}^{\mathbb{R}}: \mathcal{C} \rightarrow \mathbf{RoeCrs}$$

is fully faithful. As $[\text{Disc}]: \mathbf{RoeCrs} \rightarrow \mathbf{Crs}$ is also fully faithful (Proposition 4.3.7), the resulting composition

$$[O_{\text{top}}] := [\text{Disc}] \circ [O_{\text{top}}^{\mathbb{R}}]: \mathcal{C} \rightarrow \mathbf{Crs}$$

is again fully faithful. Note that

$$[O_{\text{top}}] = \text{Quot} \circ O_{\text{top}},$$

where $O_{\text{top}} := O_{\text{top}}^{\mathbb{R}}: \mathcal{C} \rightarrow \mathbf{RoePCrs}$ (a coarsely invariant functor).

Definition 5.2.5 (see, e.g., [10, §6.2]). For any second countable, compact space M , the *continuously controlled open cone* on M is the coarse space

$$O_{\text{top}}M := |\mathbb{R}_+ \times M|_M.$$

We saw above that $M \mapsto O_{\text{top}}M$ is a coarsely invariant functor from the category of second countable, compact topological spaces to the precoarse category \mathbf{PCrs} .

Remark 5.2.6 (compare [4, Thm. 1.23 and Cor. 1.24]). All continuously controlled coarse spaces can be described as cones in a natural way. That is, for any compactified space X , there is a natural coarse equivalence

$$O_{\text{top}}(\partial X) \xrightarrow{\sim} |X|_{\partial X}$$

(indeed, there is a natural Roe coarse equivalence $|\mathbb{R}_+ \times \partial X|_{\partial X}^{\mathbb{R}} \xrightarrow{\sim} |X|_{\partial X}^{\mathbb{R}}$). Thus, up to coarse equivalence, $|X|_{\partial X}$ only depends on the topology of the boundary ∂X , and not of X itself. We leave this to the reader.

Remark 5.2.7. Suppose M is a second countable, compact space, and $N \subseteq M$ is a closed subspace. There is a natural (coarse) inclusion $\iota: O_{\text{top}}N \hookrightarrow O_{\text{top}}M$ of continuously controlled open cones, hence a quotient coarse space

$$(O_{\text{top}}M)/[O_{\text{top}}N] := (O_{\text{top}}M)/[\iota](O_{\text{top}}N)$$

(see §3.12). One can check that the quotient $(O_{\text{top}}M)/[O_{\text{top}}N]$ is naturally coarsely equivalent to the continuously controlled open cone $O_{\text{top}}(M/N)$ on the topological quotient M/N .

Remark 5.2.8. The continuously controlled ray

$$|[0, 1[|_{\{1\}} \cong |\mathbb{R}_+|_* \cong |\mathbb{Z}_+|_* \cong O_{\text{top}}*$$

(where $*$ is a one-point space) is coarsely equivalent to $|\mathbb{Z}_+|_1$, i.e., a countable set with the terminal coarse structure.

5.3. Metric coarse simplices. We index our simplices in the same way as Mac Lane [17, Ch. VII §5], shifted by 1 from most topologists' indexing. That is, our n -simplices are topologists' $(n-1)$ -simplices (which have geometric dimension $n-1$) and we include the “true” 0-simplex.

Definition 5.3.1. As sets, put $\Delta_0 := \{0\}$, $\Delta_1 := \mathbb{R}_+ := [0, \infty[$, \dots , $\Delta_n := (\mathbb{R}_+)^n$, \dots . For each $n = 0, 1, 2, \dots$, let $d := d_n$ be the l^1 metric on Δ_n , i.e.,

$$d_n((x_0, \dots, x_{n-1}), (x'_0, \dots, x'_{n-1})) := |x_0 - x'_0| + \dots + |x_{n-1} - x'_{n-1}|,$$

and denote the resulting coarse space, called the *metric coarse n -simplex*, by

$$|\Delta_n| := |\Delta_n|_{\text{met}} := |\Delta_n|_{d_n}$$

(the metric coarse space defined in §5.1). We may also substitute the coarsely equivalent unital subspaces $(\mathbb{Z}_+)^n \subseteq (\mathbb{R}_+)^n$ for the Δ_n when convenient.

Note that we may replace the l^1 metric with any l^p -metric ($1 \leq p \leq \infty$), since

$$\|x\|_{\infty} \leq \|x\|_p \leq \|x\|_1 \leq n\|x\|_{\infty}$$

(for all $1 \leq p \leq \infty$, $x \in \Delta_n \subseteq \mathbb{R}^n$); all these metrics yield the same Roe coarse structure and hence the same coarse structure on Δ_n . See Proposition 5.3.5 below for a bit more about the “universality” of metric coarse simplices.

For each $n = 0, 1, 2, \dots$, $j = 0, \dots, n$, define a coarse map $\delta_j := \delta_j^n: |\Delta_n| \rightarrow |\Delta_{n+1}|$ by

$$(5.3.2) \quad \delta_j(x_0, \dots, x_{n-1}) := (x_0, \dots, x_{j-1}, 0, x_j, \dots, x_{n-1})$$

(for $n = 0$, let δ_0^0 be the inclusion). For each $n = 1, 2, 3, \dots, j = 0, \dots, n-1$, define a coarse map $\sigma_j := \sigma_j^n: |\Delta_{n+1}| \rightarrow |\Delta_n|$ by

$$(5.3.3) \quad \sigma_j(x_0, \dots, x_n) := (x_0, \dots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \dots, x_n).$$

It is easy to verify that the above maps are coarse and satisfy the *cosimplicial identities* (see, e.g., [7, I.1] or equations (11)–(13) in [17, Ch. VII §5]). Consequently, we get a functor from the *simplicial category* Δ to **PCrs**. Composing with the quotient functor yields the *metric coarse simplex functor*

$$|\Delta|_{\text{met}}: \Delta \rightarrow \mathbf{Crs};$$

for $n \in \text{Obj}(\Delta) = \{0, 1, 2, \dots\}$, $|\Delta|_{\text{met}}(n) = |\Delta_n|_{\text{met}}$.

Proceeding as standard (see, e.g., [7]), we may obtain *metric coarse realizations* of any simplicial set (since **Crs** has all colimits), get a corresponding notion of (metric coarse) “weak equivalence”, define *metric coarse singular sets* and a resulting *metric coarse singular homology*, and so on. We leave all of this to a future paper (or to the reader).

Remark 5.3.4. Mitchener has defined a related notion of *coarse n -cells* and *coarse $(n-1)$ -spheres* (and resulting *coarse CW-complexes*) [20, 21]. We will also defer the comparison of these with our coarse simplices (and resulting coarse simplicial complexes) to a future paper.

The l^1 (or any l^p , $1 \leq p \leq \infty$) metric coarse structure on a Δ_n is the minimal “good” one, in the following sense. Fix $n \geq 0$, and consider the maps

$$\delta_{j_1}^m \circ \dots \circ \delta_{j_{n-m}}^{n-1}: \Delta_m \rightarrow \Delta_n$$

for all $0 \leq m < n$. (The δ_j all topologically embed their domains as closed subspaces of their codomains, and hence the same is true of compositions of the δ_j .) Let us call the (set or topological) images of the each of the above maps a *boundary simplex* of the topological space Δ_n . We will not prove the following in full detail.

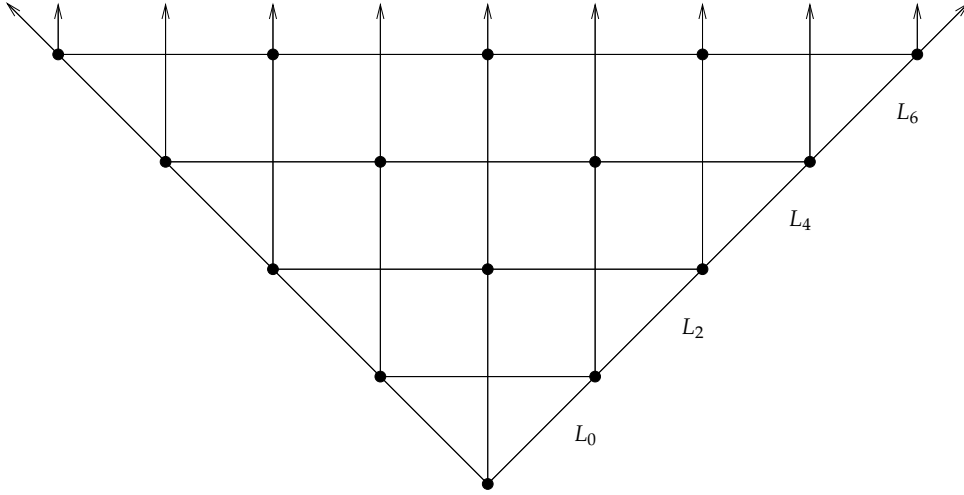
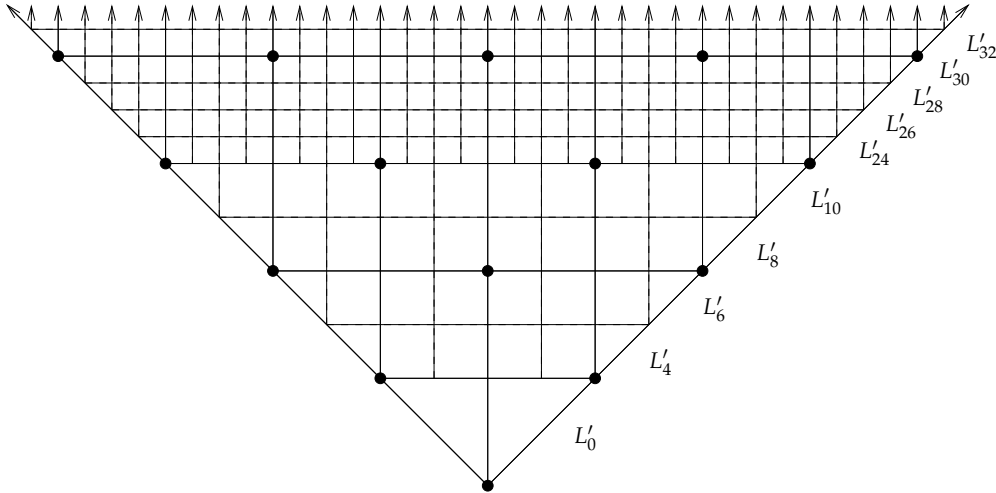
Proposition 5.3.5. *Suppose $|\Delta_n|_{\mathcal{R}}$ is a Roe coarse space with underlying topological space $\Delta_n := (\mathbb{R}_+)^n$ and Roe coarse structure \mathcal{R} . Then there is a Roe coarse map $i: |\Delta_n|_{\text{met}} \rightarrow |\Delta_n|_{\mathcal{R}}$ such that (as a set map) i maps each boundary simplex of Δ_n to itself.*

In fact, with a bit more trouble, one can even take i to be a homeomorphism. The obvious discrete version of the above, with $(\mathbb{Z}_+)^n$ in place of $\Delta_n := (\mathbb{R}_+)^n$, is rather trivial. To get a nontrivial version, one should replace $|\Delta_n|_{\mathcal{R}}$ with a “sector” which grows arbitrarily quickly away from the origin.

Sketch of proof. It is trivial for $n = 0$, so suppose that $n \geq 1$. Fix an open neighbourhood $E_0 \in \mathcal{R}$ of the diagonal 1_{Δ_n} . We will say that $B \subseteq \Delta_n$ is E_0 -bounded if $B^{\times 2} \subseteq E_0$. In the following, *disc* will mean “closed l^1 metric disc in Δ_n ”; the *diameter* of a disc will always be measured in the l^1 metric.

Tessellate Δ_n by discs diameter 1 as in Figure 5.3.6 (we illustrate the case $n = 2$), and let

$$L_{2j} := \{x \in \Delta_n: j \leq \|x\|_1 \leq j+1\}$$

FIGURE 5.3.6. The tessellation of Δ_2 by discs of l^1 -diameter 1.FIGURE 5.3.7. A refinement of the tessellation by E_0 -controlled discs.

for $j = 0, 1, 2, \dots$ be the “layers” of the tessellation. Then there is a refinement of this tessellation by discs as in Figure 5.3.7 such that each “small” disc of the refinement is E_0 -bounded; label the layers of this tessellation $L'_{2j_0}, L'_{2j_1}, \dots$ as indicated in the Figure.

Define a continuous, “tessellation preserving” map $i: \Delta_n \rightarrow \Delta_n$ which sends L_{2j_0} to L'_{2j_0} , L_{2j_1} to L'_{2j_1} , etc., collapsing the L_{2j} which do not occur in the sequence $L_{2j_0}, L_{2j_1}, \dots$; in the example illustrated in the Figures, L_2 is collapsed to the “level set” $\{x \in \Delta_2: \|x\|_1 = 1\}$, L_{12} through L_{22} are collapsed to $\{x \in \Delta_2: \|x\|_1 = 3\}$, etc.

The map i is (Roe) coarse. This map is proper and “preserves” the boundary simplices. Consider the cover $\{B_1, B_2, \dots\}$ of Δ_n by (overlapping) discs B_k of diameter 2, each a union of 2^n adjacent discs in the tessellation of Figure 5.3.6.

We have that

$$\bigcup_{k=1}^{\infty} (B_k)^{\times 2}$$

generates the Roe coarse structure of $|\Delta_n|_{\text{met}}$. The collection $\{i(B_1), i(B_2), \dots\}$ is uniformly $(E_0 \circ E_0)$ -bounded: the collection of unions of 2^n , adjacent, *equal-sized* discs in the tessellation of Figure 5.3.7 is uniformly $(E_0 \circ E_0)$ -bounded, and each $i(B_k)$ is contained in such a disc (there are four cases to check in the latter assertion: (1) i does not collapse B_k at all, (2) i completely collapses B_k , (3) i collapses the “top half” of B_k , or (4) i collapses the “bottom half” of B_k). This suffices to show that i preserves all (Roe) entourages of $|\Delta_n|_{\text{met}}$. \square

Remark 5.3.8. The above Proposition is not entirely satisfactory. $|\Delta_n|_{\text{met}}$ should satisfy a stronger universal property (which I have not yet proven): $|\Delta_n|_{\text{met}}$ should be *coarse-homotopy*-universal with the above property. That is, if S is any Roe coarse structure on Δ_n such that the above is true with $|\Delta_n|_S$ in place of $|\Delta_n|_{\text{met}}$, then $|\Delta_n|_S$ should be coarse homotopy equivalent to $|\Delta_n|_{\text{met}}$ in such a way that its boundary simplices are preserved (compare [11, Thm. 7.3]).

5.4. Continuously controlled coarse simplices. If the previously defined metric coarse structure on a simplex Δ_n is the minimal “good” one, the continuously controlled coarse structure on Δ_n defined below is the maximal “good” one (again, we will not make this precise in this paper).

For $n = 1, 2, 3, \dots$, let $\bar{\Delta}_n$ be the obvious compactification of the topological space $\Delta_n := (\mathbb{R}_+)^n$ by the standard topological simplex of geometric dimension $n - 1$. Alternatively (and equivalently, for our purposes), put

$$\Delta_n := \left\{ (x_0, \dots, x_{n-1}) \in (\mathbb{R}_+)^n : \sum_{j=0}^{n-1} x_j < 1 \right\} \quad \text{and}$$

$$\bar{\Delta}_n := \left\{ (x_0, \dots, x_{n-1}) \in (\mathbb{R}_+)^n : \sum_{j=0}^{n-1} x_j \leq 1 \right\},$$

so that $\partial\Delta_n := \bar{\Delta}_n \setminus \Delta_n$ really is the standard topological $(n - 1)$ -simplex. Put $\Delta_0 := \{0\}$ which is compact, so $\bar{\Delta}_0 = \Delta_0$ and $\partial\Delta_0 = \emptyset$.

Definition 5.4.1. For $n = 0, 1, 2, \dots$, the *continuously controlled coarse n -simplex* is the continuously controlled coarse space

$$|\Delta_n| := |\Delta_n|_{\text{top}} := |\Delta_n|_{\partial\Delta_n}.$$

Equivalently (see Remark 5.2.6), we can define $|\Delta_n|_{\text{top}}$ to be the continuously controlled open cone $O_{\text{top}}(\partial\Delta_n)$ (with underlying set $\mathbb{R}_+ \times (\partial\Delta_n)$).

Again, as in §5.3, we can define various coarse maps δ_j and σ_j between the continuously controlled coarse simplices. Indeed (using either of the above descriptions of the Δ_n), we may define them using the same formulæ (5.3.2) and (5.3.3), and hence they also satisfy the cosimplicial identities. Consequently, we get a *continuously controlled coarse simplex functor*

$$|\Delta|_{\text{top}} : \mathbf{\Delta} \rightarrow \mathbf{Crs},$$

and everything that comes along with it: *continuously controlled coarse realizations* of simplicial sets, a notion of (continuously controlled coarse) “weak equivalence”, *continuously controlled coarse singular sets* and *homology*, etc.

Remark 5.4.2. If $X = O_{\text{top}}M$ for a second countable compact topological space M (where $O_{\text{top}}M$ is the continuously controlled open cone on M from Def. 5.2.5), then it is easy to see that the continuously controlled coarse singular homology of $O_{\text{top}}M$ is exactly the singular homology of M (in this case, we would want to discard our 0-simplices and shift our indexing to match the topologists’). Continuously controlled coarse simplices have another nice feature: $|\Delta_1|_{\text{top}}$ is the continuously controlled ray, which is coarsely equivalent to $|\mathbb{Z}_+|_1$, so $|\Delta_1|_{\text{top}}$ is a product identity for most coarse spaces which arise in practice (those in $\mathbf{CrS}_{\leq |\Delta_1|_{\text{top}}}$, which includes all those which are coarsely equivalent to countable coarse spaces). However, continuously controlled simplices have a fundamental problem: they are too coarse, and so many coarse spaces X of interest (e.g., metric coarse spaces) do not even admit a coarse map $|\Delta_1|_{\text{top}} \rightarrow X$.

5.5. σ -coarse spaces and σ -unital coarse spaces. In [5, §2], Emerson–Meyer consider increasing sequences of coarse spaces. Their coarse spaces are equipped with topologies and are connected and unital (i.e., are *Roe coarse spaces* in the terminology of §4.1). We will simply handle the discrete case. (This is perhaps at significant loss of generality, since in a sense Emerson–Meyer are largely interested in “non-locally-compact coarse spaces” which we do not really examine in this paper; see Remark 4.2.3.) For our purposes, we may safely discard the connectedness assumption, though we still need unitality.

Definition 5.5.1 ([5, §2]). A (discrete) σ -coarse space (X_m) is a nondecreasing sequence

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

of unital coarse spaces such that, for all $m \geq 0$, X_m is a coarse subspace of X_{m+1} (i.e., is a subset and has the subspace coarse structure).

Remark 5.5.2. Given a sequence (X_m) which is a σ -coarse space in the sense of Emerson–Meyer (i.e., each X_m is a Roe coarse space thus may have nontrivial topology), one can obtain a nondecreasing sequence of coarse spaces by applying our discretization functor Disc to each X_m . However, $\text{Disc}(X_m)$ is typically not unital. It may be interesting to remove the unitality assumption from the above Definition, and thus be able to consider $(\text{Disc}(X_m))$ as a “nonunital σ -coarse space”.

Until otherwise stated (near the end of this section), (X_m) and (Y_n) will always denote σ -coarse spaces.

Definition 5.5.3 ([5, §4]). A coarse map $(f_n): (Y_n) \rightarrow (X_m)$ of σ -coarse spaces is a map of directed systems in \mathbf{PCrs} (taken modulo cofinality).

That is, a coarse map $(f_n): (Y_n) \rightarrow (X_m)$ is represented by a sequence of coarse maps $f_n: Y_n \rightarrow X_{m(n)}$, $n = 0, 1, \dots$, (where $0 \leq m(0) \leq m(1) \leq \cdots$ is a

nondecreasing sequence) such that the obvious diagram commutes (in **PCrs**, not modulo closeness); two representative sequences $(f_n), (f'_n)$ are considered to be equivalent if, for all n , the compositions

$$(5.5.4) \quad Y_n \xrightarrow{f_n} X_{m(n)} \hookrightarrow X_{\max\{m(n), m'(n)\}} \quad \text{and} \quad Y_n \xrightarrow{f'_n} X_{m'(n)} \hookrightarrow X_{\max\{m(n), m'(n)\}}$$

are equal.

Actually, Emerson–Meyer consider maps $\bigcup_n Y_n \rightarrow \bigcup_m X_m$, i.e., maps between set colimits which restrict to give sequences of coarse maps. This is equivalent to our definition (which avoids set colimits).

Definition 5.5.5 ([5, §4]). Coarse maps $(f_n), (f'_n): (Y_n) \rightarrow (X_m)$ are *close* if, for all n (and any, hence all, representative sequences $(f_n), (f'_n)$, respectively), the compositions (5.5.4) are close. We denote the *closeness* (equivalence) *class* of (f_n) by $[f_n]$.

Equivalently, coarse maps $(f_n), (f'_n)$ are close if they yield maps of directed systems in **Crs** which are equivalent modulo cofinality.

Since the system $X_0 \rightarrow X_1 \rightarrow \cdots$ consists of inclusion maps, the precoarse colimit **PCrs**-Colim X_m exists; one may take it to be

$$X := \mathbf{PCrs}\text{-Colim } X_m := \bigcup_m X_m$$

as a set, with coarse structure

$$\mathcal{E}_X := \langle \mathcal{E}_{X_m} : m = 0, 1, \dots \rangle_X$$

generated by the coarse structures of all the X_m . In fact, since X_m is a coarse subspace of X_{m+1} for all m ,

$$\mathcal{E}_X = \bigcup_m \mathcal{E}_{X_m}$$

(and X_m is a subspace of X); conversely, we get, for each m , that $\mathcal{E}_{X_m} = \mathcal{E}_X|_{X_m}$.

Until otherwise stated, let X be as above and similarly $Y := \mathbf{PCrs}\text{-Colim } Y_m := \bigcup_n Y_n$.

The coarse colimit **Crs**-Colim X_m also exists (since all colimits in **Crs** exist), and maps canonically to X in **Crs**. The following is easy to show.

Proposition 5.5.6. **Crs**-Colim $X_m = \mathbf{PCrs}\text{-Colim } X_m =: X$. *More precisely, the canonical arrow*

$$\mathbf{Crs}\text{-Colim } X_m \rightarrow \mathbf{PCrs}\text{-Colim } X_m =: X$$

*is an isomorphism (in **Crs**).*

By definition, any coarse map $(f_n): (Y_n) \rightarrow (X_m)$ of σ -coarse spaces yields a well-defined coarse map $f: Y \rightarrow X$. (Of course, f is just, as a set map, given by $f(y_n) := f_n(y_n)$ for all n and $y_n \in Y_n$.) Likewise, its closeness class $[f_n]$ yields a well-defined closeness class $[f]: Y \rightarrow X$.

Let \mathcal{PS} be the category of σ -coarse spaces and coarse maps, and \mathcal{S} be the category of σ -coarse spaces and closeness classes of coarse maps. We have defined functors

$$\mathcal{L} := \mathbf{PCrs}\text{-Colim}: \mathcal{PS} \rightarrow \mathbf{PCrs} \quad \text{and} \quad [\mathcal{L}] := \mathbf{Crs}\text{-Colim}: \mathcal{S} \rightarrow \mathbf{Crs}.$$

Proposition 5.5.7. *The functor $\mathcal{L}: \mathcal{PS} \rightarrow \mathbf{PCrs}$ is fully faithful.*

(Recall that “faithful” does not require injectivity on object sets!)

Proof. Faithfulness: Clear, since representative sequences $(f_n), (f'_n): (Y_n) \rightarrow (X_m)$ are cofinally equivalent if and only if they are equal on colimits (i.e., $f = f'$).

Fullness: To show that \mathcal{L} maps $\text{Hom}_{\mathcal{PS}}((Y_n), (X_m))$ to $\text{Hom}_{\mathbf{PCrs}}(Y, X)$ surjectively, we must use the unitality of the Y_n . Suppose $f: Y \rightarrow X$ is a coarse map (not a priori in the image of \mathcal{L}). For each n , Y_n is a unital subspace of Y , and hence $f(Y_n)$ is a unital subspace of X . Then $1_{f(Y_n)}$ must be an entourage of some X_m ; let $m(n)$ be the least such m . Since $\mathcal{E}_{X_{m(n)}} = \mathcal{E}_X|_{X_{m(n)}}$, $f_n := f|_{Y_n}^{X_{m(n)}}: Y_n \rightarrow X_{m(n)}$ is a coarse map. It follows that (f_n) is a coarse map of σ -coarse spaces, and that $\mathcal{L}((f_n)) = f$. \square

The following shows that unitality of the Y_n really is needed for fullness.

Example 5.5.8. Put, for each m , $X_m := |\{0, \dots, m-1\}|_1$, so that $X := \mathcal{L}((X_m))$ is just \mathbb{Z}_+ as a set, with entourages the finite subsets of $(\mathbb{Z}_+)^{\times 2}$. Put, for all n , $Y_n := X$, so that colimit $Y := X$ is nonunital ((Y_n) is not a σ -coarse space in our terminology). The identity map $Y \rightarrow X$ is coarse, but its image is not contained in any single X_m so is no “coarse map” $(f_n): (Y_n) \rightarrow (X_m)$ which yields f .

A σ -coarse space (X_m) includes as a part of its structure the “filtration” $X_0 \subseteq X_1 \subseteq \dots$. However, the particular choice of “filtration” is not important, since maps of σ -coarse spaces are taken modulo cofinality.

Corollary 5.5.9. *If $X := \mathcal{L}((X_m))$ is isomorphic in \mathbf{PCrs} to $Y := \mathcal{L}((Y_n))$ (i.e., there is a bijection of sets $f: Y \rightarrow X$ such that f and f^{-1} are both coarse maps), then (X_m) is isomorphic to (Y_n) in \mathcal{PS} (in particular, this is the case when $X = Y$ as coarse spaces).*

The situation modulo closeness parallels the above.

Proposition 5.5.10. *The functor $[\mathcal{L}]: \mathcal{S} \rightarrow \mathbf{Crs}$ is fully faithful.*

Proof. Faithfulness: Since each Y_n is a subspace of Y and each X_m a subspace of X , closeness of $f = \mathcal{L}((f_n))$ to $f' = \mathcal{L}((f'_n))$ implies closeness of the compositions (5.5.4) (noting that $f_n = f|_{Y_n}^{X_{m(n)}}$ and similarly for f'_n).

Fullness: Here, we implicitly use the unitality condition. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{PS} & \xrightarrow{\mathcal{L}} & \mathbf{PCrs} \\ \text{Quot} \downarrow & & \downarrow \text{Quot} \\ \mathcal{S} & \xrightarrow{[\mathcal{L}]} & \mathbf{Crs} \end{array} \quad .$$

Since \mathcal{L} is full and evidently the quotient functors are also full and map surjectively onto object sets, $[\mathcal{L}]$ is full. \square

It is not clear to me whether fullness of $[\mathcal{L}]$ fails if the unitality condition is removed from Definition 5.5.5; the counterexample of Example 5.5.8 fails.

Corollary 5.5.11. *If $X := \mathcal{L}((X_m))$ is coarsely equivalent (i.e., isomorphic in \mathbf{Crs}) to $Y := \mathcal{L}((Y_n))$, then (X_m) is isomorphic to (Y_n) in \mathcal{S} .*

It follows from Propositions 5.5.10 and 5.5.7 that \mathcal{L} and $[\mathcal{L}]$ are equivalences (of categories) onto their images. We now consider what the images of these functors are (and how one constructs “inverse” functors).

Let us “reset” our notation: X, Y are just coarse spaces, not necessarily coming from σ -coarse spaces, and $(X_m), (Y_n)$ are not assumed to have any meaning.

Definition 5.5.12. A coarse space X is σ -unital if there is a nondecreasing sequence

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

of unital subspaces of X such that each unital subspace $X' \subseteq X$ is contained in some X_m (m depending on X').

It is implied that $X = \bigcup_m X_m$, though this equality certainly does not imply that each unital subspace of X is contained in some X_m .

Let $\mathbf{PCrs}_\sigma \subseteq \mathbf{PCrs}$ and $\mathbf{Crs}_\sigma \subseteq \mathbf{Crs}$ denote the full subcategories of σ -unital coarse spaces. Clearly, \mathcal{L} and $[\mathcal{L}]$ map both into and onto \mathbf{PCrs}_σ and \mathbf{Crs}_σ , respectively. We get the following.

Theorem 5.5.13. *The functors $\mathcal{L}: \mathcal{PS} \rightarrow \mathbf{PCrs}_\sigma$ and $[\mathcal{L}]: \mathcal{S} \rightarrow \mathbf{Crs}_\sigma$ are equivalences of categories.*

It is also easy to construct “inverse” functors. Choose, for each σ -unital X , a “filtration” (X_m) . Then $X \mapsto (X_m)$ (and, for $f: Y \rightarrow X$, $f \mapsto (f_n)$, where f_n is an appropriate range restriction of $f|_{Y_n}$) gives a functor “inverse” to $\mathcal{L}: \mathcal{PS} \rightarrow \mathbf{PCrs}_\sigma$. Choosing representative coarse maps, one does the same to obtain an “inverse” to $[\mathcal{L}]: \mathcal{S} \rightarrow \mathbf{Crs}_\sigma$.

5.6. Quotients and Roe algebras. We shall assume that the reader is familiar with the definition and construction of the Roe algebras $C^*(X)$ for X a (Roe) coarse space (see, e.g., [9]); the generalization to our nonunital situation is straightforward. We will follow the standard, abusive practice of pretending that $X \mapsto C^*(X)$ is a functor. (The situation is slightly complicated by our nonunital situation. However, there are a number of ways of obtaining an actual functor, just not to the category of C^* -algebras. One could, for example, construct a coarsely invariant functor from \mathbf{Crs} to the category of C^* -categories [19].) The important fact is that, applying K -theory, one gets a coarsely invariant functor $X \mapsto K_\bullet(C^*(X))$. The following should be regarded as a sketch, with more details to follow in a future paper.

Fix a coarse space X and a subspace $Y \subseteq X$, and denote the inclusion $Y \hookrightarrow X$ by ι . We note that the following does not depend on our generalizations, and even works in the “classical” unital context; if X is a Roe coarse space in the sense of §4, Y should be closed in X . Recall that we simply denote the quotient $X/[\iota](Y)$ (defined in §3.12) by $X/[Y]$. The coarse space $X/[Y]$ is easy to describe: It is just X as a set, with coarse structure generated by the entourages of X and

those of $\text{Term}(Y)$ (if X is unital, the latter are just those of the terminal coarse structure on Y).

The quotient $Y/[Y] \text{Term}(Y)$ is a subspace of $X/[Y]$, with $\iota: Y/[Y] \hookrightarrow X/[Y]$ an inclusion. We get a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & X \\ \tilde{q} \downarrow & & \downarrow q \\ Y/[Y] & \xrightarrow{\iota} & X/[Y] \end{array},$$

where \tilde{q} and q represent the quotient maps (which one can take to be identity set maps). This square gives rise to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_X^*(Y) & \xrightarrow{\iota_*} & C^*(X) & \longrightarrow & Q_{X,Y} \longrightarrow 0 \\ & & \tilde{q}_* \downarrow & & q_* \downarrow & & q_* \downarrow \\ 0 & \longrightarrow & C_{X/[Y]}^*(Y/[Y]) & \xrightarrow{\iota_*} & C^*(X/[Y]) & \longrightarrow & Q_{X/[Y], Y/[Y]} \longrightarrow 0 \end{array}$$

of C^* -algebras whose rows are exact; $C_X^*(Y)$ denotes the ideal of $C^*(X)$ of operators supported near Y (which can be identified with $C^*(\text{Coim}[\iota])$, where $\text{Coim}[\iota] = X$ as a set with the nonunital coarse structure of entourages of X supported near Y ; see Def. 3.10.9) and $Q_{X,Y}$ is the quotient C^* -algebra (and similarly for the second row).

Next, one observes that q_* is an isomorphism of C^* -algebras hence induces an isomorphism on K -theory. Let us specialize to the case when X is unital (from which it follows that Y and the quotient coarse spaces are also unital), and examine the consequences. If Y is finite (or compact, in the Roe coarse space version) then $X = X/[Y]$ and $Y = Y/[Y]$, so \tilde{q}_* and q_* are identity maps on the level of C^* -algebras and hence the diagram is trivial.

On the other hand, if Y is infinite, then $Y/[Y] = |Y|_1$ has the terminal coarse structure and one can show by a standard “Eilenberg swindle” (see, e.g., [10, Lem. 6.4.2]) that $K_\bullet(C_{X/[Y]}^*(Y/[Y])) = 0$. Thus we get a canonical isomorphism

$$K_\bullet(C^*(X/[Y])) \xrightarrow{\sim} K_\bullet(Q_{X/[Y], Y/[Y]}) \xrightarrow{\sim} K_\bullet(Q_{X,Y})$$

on K -theory. Consequently, using the isomorphism $K_\bullet(C^*(Y)) \xrightarrow{\sim} K_\bullet(C_X^*(Y))$ (which is easy to prove under most circumstances), we get a long (or six-term) exact sequence

$$(5.6.1) \quad \cdots \xrightarrow{\partial} K_\bullet(C^*(Y)) \rightarrow K_\bullet(C^*(X)) \rightarrow K_\bullet(C^*(X/[Y])) \xrightarrow{\partial} K_{\bullet-1}(C^*(Y)) \rightarrow \cdots$$

Remark 5.6.2 (continuous control). In the above situation, suppose that $X = OM$ and $Y = ON$ are continuously controlled open cones, where N is a nonempty closed subspace of a second countable, compact space M and we abbreviate $O := O_{\text{top}}$. Then there are natural isomorphisms

$$(5.6.3) \quad K_\bullet(C^*(OM)) \cong \tilde{K}^{1-\bullet}(C(M)) = \tilde{K}_{\bullet-1}(M) \quad \text{and} \quad K_\bullet(C^*(ON)) \cong \tilde{K}_{\bullet-1}(N),$$

where \tilde{K} is reduced K -homology (see, e.g., [10, Cor. 6.5.2]). One can check that there is also a natural isomorphism

$$K_{\bullet}(C^*(OM/[ON])) \cong K_{\bullet-1}(M, N)$$

(to relative K -homology), so that the above long exact sequence (5.6.1) naturally maps isomorphically to the reduced K -homology sequence

$$\cdots \xrightarrow{\partial} \tilde{K}_{\bullet-1}(N) \rightarrow \tilde{K}_{\bullet-1}(M) \rightarrow K_{\bullet-1}(M, N) \xrightarrow{\partial} K_{\bullet-2}(N) \rightarrow \cdots$$

We have three natural isomorphisms

$$\begin{aligned} K_{\bullet}(C^*(OM/[ON])) &\cong K_{\bullet}(C^*(O(M/N))), \\ K_{\bullet}(C^*(O(M/N))) &\cong \tilde{K}_{\bullet-1}(M/N), \quad \text{and} \\ K_{\bullet-1}(M, N) &\cong \tilde{K}_{\bullet-1}(M/N), \end{aligned}$$

from Remark 5.2.7, as in (5.6.3) above, and by excision for K -homology, respectively; these are mutually compatible in the obvious sense.

Example 5.6.4 (K -theory of $O_{\text{top}}S^n$). We give yet another version of a standard calculation (see, e.g., [10, Thm. 6.4.10]). For $n \geq 0$, denote the topological n -sphere by S^n and, for $n \geq 1$, the closed n -disc by D^n ; recall that D^n has “boundary” S^{n-1} and that $D^n/S^{n-1} \cong S^n$. Again we abbreviate $O := O_{\text{top}}$.

First, we compute the K -theory of $X := OS^0$. Put $Y := O\{-1\} \subseteq X$, $X' := O\{1\} \subseteq X$, and $Y' := \{0\} \subseteq Y \cap X'$. It is well known that $K_{\bullet}(C^*(X')) = 0$ and $K_{\bullet}(C_X^*(Y)) = 0$ (by the aforementioned “Eilenberg swindle”), and that

$$K_{\bullet}(C_{X'}^*(Y')) = \begin{cases} \mathbb{Z} & \text{if } \bullet \equiv 0 \pmod{2}, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

(since $C_{X'}^*(Y')$ is just the compact operators). We have a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{X'}^*(Y') & \longrightarrow & C^*(X') & \longrightarrow & Q' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_X^*(Y) & \longrightarrow & C^*(X) & \longrightarrow & Q \longrightarrow 0 \end{array}$$

But one checks easily that the map $Q' \rightarrow Q$ is an isomorphism of C^* -algebras, hence from the K -theory long exact sequences we get

$$\begin{aligned} K_{\bullet}(C^*(OS^0)) &= K_{\bullet}(Q) = K_{\bullet}(Q') \\ &= K_{\bullet-1}(C_{X'}^*(Y')) = \begin{cases} \mathbb{Z} & \text{if } \bullet \equiv 1 \pmod{2}, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We proceed to calculate the K -theory of OS^n , $n \geq 1$, by induction. Put $X := OD^n$ and $Y := OS^{n-1} \subseteq X$, and recall that $X/[Y] = O(D^n/S^{n-1}) = OS^n$. Then, by Remark 5.6.2 above, we have a long exact sequence

$$\cdots \xrightarrow{\partial} K_{\bullet}(C^*(Y)) \rightarrow K_{\bullet}(C^*(X)) \rightarrow K_{\bullet}(C^*(OS^n)) \xrightarrow{\partial} K_{\bullet-1}(C^*(Y)) \rightarrow \cdots$$

By another “Eilenberg swindle”, one shows that $K_\bullet(C^*(X)) = 0$ and hence

$$K_\bullet(C^*(OS^n)) = K_{\bullet-1}(C^*(Y)) = \begin{cases} \mathbb{Z} & \text{if } \bullet \equiv n-1 \pmod{2}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.6.5 (suspensions in K -homology). Suppose that A is a separable C^* -algebra. It is known that, for $n \geq 0$, elements of the Kasparov K -homology group $K^{n+1}(A)$ can be represented by (equivalence classes of) C^* -algebra morphisms

$$(5.6.6) \quad \varphi: A \rightarrow C^*(OS^n)$$

(see [16, 26]; I caution that, in my opinion, this is probably not the “best” coarse geometric description of K -homology, but work remains ongoing). The pairing of $K_m(A)$ with a K -homology class represented by such φ is given simply by applying K -theory to φ (and using the computation as in the previous Example).

Fix $n \geq 1$ and suppose that we are given an element of $K^n(\Sigma A)$, where $\Sigma A := C_0([0, 1]) \otimes A$ is the C^* -algebraic suspension of A , represented by a morphism

$$(5.6.7) \quad \tilde{\psi}: \Sigma A \rightarrow C^*(OS^{n-1}).$$

Actually, let us assume something stronger, that we are given a morphism ψ which fits into the following commutative diagram whose *rows* are with exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma A & \longrightarrow & CA & \longrightarrow & A & \longrightarrow & 0 \\ & & \psi|_{\Sigma A} \downarrow & & \psi \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_X^*(Y) & \longrightarrow & C^*(X) & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{X/[Y]}^*(Y/[Y]) & \longrightarrow & C^*(X/[Y]) & \longrightarrow & Q' & \longrightarrow & 0 \end{array},$$

where $CA := C_0([0, 1]) \otimes A$ is the cone on A , $X := OD^n$, and $Y := OS^{n-1} \subseteq X$. (In fact, given a $\tilde{\psi}$, one can find a ψ such that

$$\begin{array}{ccc} K_\bullet(\Sigma A) & \xrightarrow{\tilde{\psi}} & K_\bullet(C^*(Y)) \\ = \downarrow & & \sim \downarrow \\ K_\bullet(\Sigma A) & \xrightarrow{\psi|_{\Sigma A}} & K_\bullet(C_X^*(Y)) \end{array}$$

commutes. This is not easy to prove, and seems to require that A be separable.)

Denote the composition $A \rightarrow Q \rightarrow Q'$ by φ . From the previous Example, we have natural isomorphisms

$$K_\bullet(OS^{n-1}) = K_\bullet(C_X^*(Y)) = K_{\bullet+1}(Q) = K_{\bullet+1}(Q') = K_{\bullet+1}(X/[Y]) = K_{\bullet+1}(OS^n).$$

Moreover, since $K_\bullet(CA) = 0$, we have $K_\bullet(\Sigma A) = K_{\bullet+1}(A)$. These isomorphisms are all compatible, in the sense that ψ and φ are naturally equivalent on K -theory (with a dimension shift).

In fact, one can “lift” the morphism φ to a morphism $\varphi: A \rightarrow C^*(X/[Y]) = C^*(OS^n)$ in the weak sense that the composition $A \xrightarrow{\varphi} C^*(X/[Y]) \rightarrow Q'$ is equal to φ on the level of K -theory. (This is not too difficult, but again seems to require

that A be separable.) This provides a map from the K -homology group $K^n(\Sigma A)$ (described as classes of morphisms as in (5.6.7)) to the group $K^{n+1}(A)$ (described as in (5.6.6)).

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